Hydrodynamic representation and energy balance for Dirac and Weyl fermions in curved space-times

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Using a generalized Madelung transformation, we derive the hydrodynamic representation of the Dirac equation in arbitrary curved space-times coupled to an electromagnetic field. We obtain Dirac-Euler equations for fermions involving a continuity equation and a first integral of the Bernoulli equation. Using the comparison of the Dirac and Klein-Gordon equations we obtain the balance equation for fermion particles. We also use the correspondence between fermions and bosons to derive the hydrodynamic representation of the Weyl equation which is a chiral form of the Dirac equation.

fermions | hydrodynamic representation | Madelung transformation

1. Introduction

The Standard Model of elementary particles establishes that there exist two kinds of particles, fermions and bosons. In previous works [1, 2], the energy balance for bosons was derived starting from the general relativistic Klein-Gordon (KG) equation. In the present work, we study a system of fermions described by the Dirac equation in arbitrary curved space-times taking into account electromagnetic effects. We also use the Weyl equation which is a chiral form of the Dirac equation due to the relationship between the Lie algebras of the symmetry groups for both systems of particles. We give the hydrodynamic representation of the Dirac and Weyl equations for fermions using previous results obtained for boson particles. This representation is built analogously as in quantum mechanics (QM) and as in the bosonic case [1], where it was introduced by the Madelung transformation in order to find an alternative interpretation of a boson system. This interpretation has been very useful in astrophysics [2]. In this article, we extend the previous transformation to the fermionic case, in the same way we pretend to give an alternative interpretation of the femionic systems.

Many examples of fermion particles in strong gravitational fields can be found in nature. Indeed, the curvature of space-time plays an important role in a neutron star, in the early Universe, or in a fermion cloud (e.g. a dark matter halo) in the vicinity of a black hole. We need to develop a general framework to identify what are the different energy contributions in such systems. In this work we use the geometrical decomposition of the metric in 3+1 slices and the tetrad formalism to study the particle spin in an arbitrary space-time. We define the gamma matrices in curved space-times and derive the generalized Dirac and Weyl equations. Then, using the Madelung transformation, we introduce a hydrodynamic representation of the Dirac and Weyl spinors. This hydrodynamic representation can help us to describe the fermionic system in a general framework. We can highlight that this description is convenient because it is easier to make a physical interpretation, since the hydrodynamic representation is given in some variable such as number of particles, speed, potential or energy. In fact, a non-obvious result is the energy balance equation, which is the first law of thermodynamics, which comes from the Dirac equation with the Madelung transformation for spinors. Although the equations obtained from this representation are more complicated than in the usual way, it can help us to have a closer answer for interpretations of quantum theory, for example, the de Broglie-Bohm interpretation[3–5]. In addition, we can compare the hydrodynamics and energy balance in different frames for classical and quantum particles, as well as spin and spinless particles, such as bosons and fermions.

2. Field Equations

A. Dirac representation. We start using the tetrad formalism for the space-time geometry, and the canonical expansion of the space-time in a 3+1 ADM decomposition [6–11], such that the coordinate t is the parameter of evolution. The 3+1 metric reads

$$\mathrm{d}s^2 = N^2c^2\mathrm{d}t^2 - h_{ij}\left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j c\,\mathrm{d}t\right), \qquad [1]$$

where N represents the lapse function which measures the proper time of the observers traveling along the world line, N^i is the shift vector and h_{ij} is the 3-dimensional slice-metric. We write eq.(1) in the tetrad formalism as $\mathrm{d}s^2 = \eta_{ab}e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu$, where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. Here $e^a = e^a{}_\mu dx^\mu$ is the set of one-forms base of the cotangent space at the space-time manifold given by

$$\begin{aligned} e^0 &= Nc\mathrm{d}t, \\ e^k &= \hat{e}^k{}_i\left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right), \end{aligned} \qquad [2]$$

with inverse given by $\hat{e}^k = \hat{e}^k{}_i \mathrm{d}x^i$ the one-form base to the three-dimensional slice of the cotangent manifold, such that $h_{ij} = \delta_{kl}\hat{e}^k{}_i\hat{e}^l{}_j$. We can also define the set of vectors base of the tangent-space to the space-time as $e_a = e_a{}^\mu\partial_\mu$, such that $e^a e_b = \delta^a_b$. The action of a fermion system in curved space-times coupled to an electromagnetic field A_μ is given by $S\left[\psi(x^\mu), \partial_\mu\psi(x^\mu)\right] = \int \mathcal{L}\left(\psi(x^\mu), \partial_\mu\psi(x^\mu)\right) d^4x$, where $\mathcal{L} = \mathcal{L}\left(\psi(x^\mu), \partial_\mu\psi(x^\mu)\right)$ is the Lagrangian density [12–14]:

$$\mathcal{L} = \sqrt{-g}\frac{i\hbar c}{2}\left[\psi^\dagger B\gamma^\mu\left(D_\mu\psi\right) - \left(D_\mu\psi\right)^\dagger B\gamma^\mu\psi + \frac{2imc}{\hbar}\psi^\dagger B\psi\right]. \qquad [3]$$

Here, $D_\mu = \nabla_\mu + \frac{iq}{\hbar c}A_\mu$ is the total covariant derivative accounting for electromagnetic effects. The covariant derivative of a spinor $\psi = (\psi_{\dot\nu})$ is given by $\nabla_\mu(\psi_{\dot\nu}) = \partial_\mu(\psi_{\dot\nu}) + \Gamma^{\dot\alpha}_{\mu\dot\nu}(\psi_{\dot\alpha})$,

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where $\Gamma^{\dot{\alpha}}_{\mu\dot{\nu}}$ is the spin connection[6, 15]. Observe the internal indices as dot indices. Using the least action principle it is possible to obtain from eq.(3) the corresponding Dirac equation. This equation is given by

$$[i\hbar\gamma^{\mu}(\nabla_{\mu}+iqA_{\mu})-mc]\,\psi=0, \qquad [4]$$

where $\hbar$, c are the Planck constant and the speed of light respectively, while q, m are the charge and mass of the fermion particle and ψ is its spinor. Besides, the gamma matrices γ^{μ} are related to the spin and space-time geometry. They can be written as $\gamma^{\mu}=e^{\mu}{}_{a}\tilde{\gamma}^{a}$, where $\tilde{\gamma}^{a}$ are the gamma matrices in flat space-time, which are well-know from standard Quantum Field Theory (QFT) [16–18] Henceforth, we use the natural units. Therefore,

$$\begin{aligned}\gamma^{0} &= N\tilde{\gamma}^{0},\\ \gamma^{k} &= \hat{e}^{k}{}_{j}(\tilde{\gamma}^{j}+N^{j}\tilde{\gamma}^{0}).\end{aligned} \qquad [5]$$

In general, these matrices fulfill the following anti-commutation relation[6, 19]

$$\{\gamma^{\mu},\gamma^{\nu}\}=\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}\mathbb{I}, \qquad [6]$$

where $g_{\mu\nu}$ represents the metric that describes the space-time geometry. On the other hand, we observe that, in general, the gamma matrices obey the following relation[12–14, 20]

$$(\gamma^{\mu})^{\dagger}=B\gamma^{\mu}B^{-1}, \qquad [7]$$

where B is a hermitian matrix, i.e. $B^{\dagger}=B$, that is uniquely determined by the gamma matrices γ^{μ}. As usual, we denote by $B^{\dagger}$ the conjugate (or Hermitian) transpose of B.
Furthermore, we note that in QFT the relation (7) is fulfilled when $B=\tilde{\gamma}^{0}$ and the gamma matrices are in flat space-time. From the action (3) of the fermion system we can find the equation for the transpose conjugated spinor by making an infinitesimal variation of this action with respect to ψ that is given by

$$i\left(\nabla_{\mu}\bar{\psi}\right)\gamma^{\mu}-i\psi^{\dagger}\nabla_{\mu}\left(B\gamma^{\mu}\right)+i\bar{\psi}\nabla_{\mu}\gamma^{\mu}+\bar{\psi}A_{\mu}\gamma^{\mu}+m\bar{\psi}=0. \qquad [8]$$

We denote the adjoint spinor as $\bar{\psi}=\psi^{\dagger}B$. Using the gamma matrices in flat space-time and the fact that $B=\tilde{\gamma}^{0}$, we recover the definition of $\bar{\psi}$ in QFT. However, in an arbitrary space-time $\nabla_{\mu}\gamma^{\mu}$ is distinct from zero, since $\gamma^{\mu}=e^{\mu}{}_{a}\tilde{\gamma}^{a}$. Therefore, in general $\nabla_{\mu}e^{\mu}{}_{a}$ is non-zero. We can get the conserved charge from the Noether theorem [21]. The Dirac current is

$$J^{\mu}=\bar{\psi}\gamma^{\mu}\psi=\psi^{\dagger}B\gamma^{\mu}\psi. \qquad [9]$$

To obtain the continuity equation

$$\nabla_{\mu}J^{\mu}=0. \qquad [10]$$

we get

$$\nabla_{\mu}J^{\mu}=\psi^{\dagger}\nabla_{\mu}\left(B\gamma^{\mu}\right)\psi. \qquad [11]$$

If we require that the continuity equation (10) is fulfilled, i.e., that the number of particles is conserved, then we need $\nabla_{\mu}\left(B\gamma^{\mu}\right)=0$, or equivalently

$$(\nabla_{\mu}B)\gamma^{\mu}=-B\nabla_{\mu}\gamma^{\mu}. \qquad [12]$$

In references [13, 14], the authors conclude that the condition (12) is the most convenient because it is implied by $\nabla_{\mu}\gamma^{\nu}=0$ and $\nabla_{\mu}B=0$.

Since, the spinor field used is coupled to an electromagnetic field, we show the equations that describe the electromagnetic field. Thus, with the Maxwell four-potential we can define the Faraday tensor

$$F_{\mu\nu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}. \qquad [13]$$

In the electromagnetic theory, the Faraday tensor $F_{\mu\nu}$ satisfies the Maxwell field equations

$$\nabla_{\nu}F^{\nu\mu}=J^{E\mu}, \qquad [14]$$

where $J^{E\mu}$ is the four-electromagnetic current.

B. Weyl representation. The Dirac equation for 1/2-spin particles is associated with the $SO(1,3)$ symmetry group. Nevertheless, we can introduce a new representation as in standard QFT, since there exists a surjective homomorphism between the $SO(1,3)$ and $SU(2)\otimes SU(2)$ Lie groups.
In terms of the Pauli matrices σ^{μ} the 4×4 gamma matrices γ^{μ} can be written as two 2×2 block matrices

$$\gamma^{0} = N\tilde{\gamma}^{0}=N\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}, \qquad [15]$$

$$\gamma^{j} = \begin{pmatrix}0 & -\hat{e}^{j}{}_{i}(\tilde{\sigma}^{i}-N^{i}\mathbb{I})\\ \hat{e}^{j}{}_{i}(\tilde{\sigma}^{i}+N^{i}\mathbb{I}) & 0\end{pmatrix}, \qquad [16]$$

where $\tilde{\sigma}^{i}$ are the 2×2 Pauli matrices in flat space-time

$$\tilde{\sigma}^{1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \tilde{\sigma}^{2}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \tilde{\sigma}^{3}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad [17]$$

and $\mathbb{I}$ is the 2×2 identity matrix. The γ^{μ} matrices satisfy $\left(\gamma^{0}\right)^{\dagger}=\gamma^{0}$ and $\left(\gamma^{j}\right)^{\dagger}=-\gamma^{j}+2N^{j}\gamma^{0}/N$. At this point, we need to adopt the standard representation for the gamma matrices in a flat space-time $\tilde{\gamma}^{\mu}$ as follows

$$\tilde{\gamma}^{0}=\begin{pmatrix}0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix},\quad \tilde{\gamma}^{j}=\begin{pmatrix}0 & -\tilde{\sigma}^{j}\\ \tilde{\sigma}^{j} & 0\end{pmatrix}. \qquad [18]$$

This representation helps us to build the Weyl representation. Additionally, in the Weyl representation we can write a Dirac fermion as a four-spinor ψ made of two spinors, each of which having two components, for instance

$$\psi=\begin{pmatrix}\psi_{R}\\ \psi_{L}\end{pmatrix}, \qquad [19]$$

where ψ_{R} and ψ_{L} are the right- and the left- handed Weyl spinors, respectively. If we write the adjoint spinor $\bar{\psi}$ and use the Weyl representation, it follows that

$$\bar{\psi}=\psi^{\dagger}B=\left(\psi_{R}^{\dagger},\psi_{L}^{\dagger}\right)B, \qquad [20]$$

where B is the matrix from eqs. (7) and (12). If we use the relation (7) it is straightforward to see that the matrix B must have the following form

$$B=\begin{pmatrix}0 & B_{\zeta}\\ B_{\zeta} & 0\end{pmatrix}, \qquad [21]$$

where the 2×2 matrix B_{ζ} is a diagonal matrix, $B_{\zeta}=b\mathbb{I}$, with $b=b(x^{\mu})$. Adopt a specific representation for the symmetry group, which is done without loss of generality. Hence, using

the definition of the spinor and its adjoint we can write the Dirac quadricurrent J^μ from eq. (9). This yields

$$J^0 = Nb(\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L), \quad [22]$$
$$J^j = b\hat{e}^j{}_i(\psi_R^\dagger(\tilde{\sigma}^i + N^i\mathbb{I})\psi_R - \psi_L^\dagger(\tilde{\sigma}^i - N^i\mathbb{I})\psi_L). \quad [23]$$

In order to simplify the notation, we now define the vectors of 2×2 matrices $\mathbb{S}^a = (\mathbb{I}, \tilde{\sigma}^j + N^j\mathbb{I})$ and $\bar{\mathbb{S}}^a = (-\mathbb{I}, \tilde{\sigma}^j - N^j\mathbb{I})$ in terms of the Pauli matrices. $\mathbb{S}^a$ and $\bar{\mathbb{S}}^a$ are the (generalized) Pauli matrices in flat space-time. In terms of these new definitions, the density currents read

$$J^\mu = b\hat{e}^\mu{}_i(\psi_R^\dagger\mathbb{S}^i\psi_R - \psi_L^\dagger\bar{\mathbb{S}}^i\psi_L) = b(\psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L), \quad [24]$$

where we have defined the 2×2 Pauli matrices in a curved space-time by $\sigma^\mu = e^\mu{}_a\mathbb{S}^a$ and $\bar{\sigma}^\mu = e^\mu{}_a\bar{\mathbb{S}}^a$. With this definition, the matrices γ^j read

$$\gamma^j = \begin{pmatrix} 0 & -\bar{\sigma}^j \\ \sigma^j & 0 \end{pmatrix}. \quad [25]$$

Furthermore, observe that the σ^j matrices follow the same commutation relations as the flat space-time Pauli matrices. This means that $[\sigma^i, \bar{\sigma}^j] = -\hat{e}^i_k\hat{e}^j_l[\tilde{\sigma}^k, \tilde{\sigma}^l]$. We can now apply the Weyl representation to rewrite the Dirac equation (4) for a spinor with four components as

$$\begin{pmatrix} i\sigma^\mu\left(\bar{\nabla}_\mu + iqA_\mu\right)\psi_R - m\psi_L \\ i\bar{\sigma}^\mu\left(\nabla_\mu + iqA_\mu\right)\psi_L - m\psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad [26]$$

These are the Weyl equations for a spinor in a curved space-time coupled to an electromagnetic field. If we set $B = b\tilde{\gamma}^0$, the current density now reads

$$J^\mu = b\left(\psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L\right). \quad [27]$$

Explicitly, we have for the spatial part

$$J^j = b\hat{e}^j_i\left(\psi_R^\dagger\tilde{\sigma}^i\psi_R - \psi_L^\dagger\tilde{\sigma}^i\psi_L + \frac{N^i}{Nb^2}J^0\right). \quad [28]$$

On the other hand, we can be obtained from the identities

$$\gamma^\mu\gamma^\nu F_{\mu\nu}\psi = \begin{cases} (2NN^kF_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k)\psi_R \\ -(2NN^kF_{0k} - i\hat{F}_{ij}\epsilon^{ij}{}_k\tilde{\sigma}^k)\psi_L \end{cases}, \quad [29]$$

and using definition (25), we find that

$$\gamma^\mu(\nabla_\mu\gamma^\nu)(D_\nu\psi) = \begin{cases} -\bar{\mathbb{S}}^a\mathbb{S}^b(\hat{\nabla}_a\hat{e}^\nu_b)(D_\nu\psi_R) \\ -\mathbb{S}^a\bar{\mathbb{S}}^b(\hat{\nabla}_a\hat{e}^\nu_b)(D_\nu\psi_L) \end{cases} \quad [30]$$

where $\epsilon^{ij}{}_k$ is the usual Levi-Civita tensor, $\hat{F}_{ij} = \hat{e}^l_i\hat{e}^m_j F_{lm}$ is the directional Maxwell tensor $\hat{F}_{ij} = (\hat{e}^l_i\hat{\nabla}_j - \hat{e}^l_j\hat{\nabla}_i)A_l$, and $\hat{\nabla}_a = \hat{e}^\alpha_a\nabla_\alpha$ is the directional covariant derivative which defines the Cartan connection $\hat{\nabla}_c\hat{e}^\nu_b = \Gamma^a_{bc}\hat{e}^\nu_a$. The Cartan connection $\Gamma^a_{bc} = \hat{e}^a_\nu\hat{\nabla}_c\hat{e}^\nu_b$ determines the Cartan first fundamental form $d\hat{e}^a + \Gamma^a_b\wedge\hat{e}^b$ for the connections $\Gamma^a_b = \Gamma^a_{bd}\hat{e}^d$ with the property that $\Gamma_{ab} + \Gamma_{ba} = 0$, where $\Gamma_{ab} = \eta_{ad}\Gamma^d_b$.

3. Weyl Hydrodynamic Representation

We now have all the ingredients to propose a hydrodynamic representation for the Weyl fermions, following the same procedure as the one developed for the Schrödinger and KG equations in Refs.[1, 2]. We start to propose our Madelung transformation in the Weyl spinor, using the exponential map, that is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} R_R \\ R_L \end{pmatrix} e^{i\theta}. \quad [31]$$

Since ψ_R and ψ_L are two spinors, we observe that R_R and R_L are two two-dimensional vectors. The Weyl representation of the adjoint spinor $\bar{\Psi}$ when $B = b\tilde{\gamma}^0$ is

$$\bar{\Psi} = b\left(\psi_R^\dagger, \psi_L^\dagger\right)\tilde{\gamma}^0 = \left(R_R^\dagger, R_L^\dagger\right)e^{-i\theta}. \quad [32]$$

We use R_L and R_R as complex two-spinors and θ as a complex function. Therefore, using the Madelung transformation (31) in the Weyl equations (26) and applying the Lie algebra and the Lie group, we can get the following expression

$$\begin{pmatrix} -\sigma^\mu\left(\bar{\nabla}_\mu\theta\right)R_R + i\sigma^\mu\left(\bar{\nabla}_\mu R_R\right) - q\sigma^\mu A_\mu R_R \\ -\bar{\sigma}^\mu\left(\nabla_\mu\theta\right)R_L + i\bar{\sigma}^\mu\left(\nabla_\mu R_L\right) - q\bar{\sigma}^\mu A_\mu R_L \end{pmatrix} = \begin{pmatrix} mR_L \\ mR_R \end{pmatrix}. \quad [33]$$

These are the Weyl equations in curved space-time with the Madelung transformation. We can also apply the Madelung transformation (31) and (32) to the current density (27), thereby obtaining

$$J^\mu = b\left(R_R^\dagger\bar{\sigma}^\mu R_R - R_L^\dagger\sigma^\mu R_L\right). \quad [34]$$

Its components are

$$J^0 = Nb(R_R^\dagger R_R + R_L^\dagger R_L) = Nb|n|, \quad [35]$$
$$J^j = b\left(\hat{e}^j_3(|n_{\dot{1}}| - |n_{\dot{2}}| - |n_{\dot{3}}| + |n_{\dot{4}}|)\right. + 2\hat{e}^j_1(\sqrt{n_{\dot{1}}n_{\dot{2}}} - \sqrt{n_{\dot{3}}n_{\dot{4}}}) + \hat{e}^j_i N^i|n|. \quad [36]$$

We note that the zero component, where $|n| = \sum_{\dot{\nu}=\dot{1}}^{\dot{4}}|n_{\dot{\nu}}|$ is the density number of fermions in the system, gives the number of both right- and left-handed particles. We can write the following expressions $|\psi_R|^2 = \psi_R^\dagger\psi_R = R_R^\dagger R_R = |n_R|$ and $|\psi_L|^2 = \psi_L^\dagger\psi_L = R_L^\dagger R_L = |n_L|$ for the right- and left-handed spinors, as in the Dirac case. Thus, $|n_R|$, $|n_L|$ are the right- and left- handed particle number and $|n| = |n_R| + |n_L|$ is the total density number.

We apply the operator $i\gamma^\mu D_\mu = i\gamma^\mu\nabla_\mu - q\gamma^\mu A_\mu$ to the Dirac equation (4) written under the form $i\gamma^\mu\nabla_\mu\psi = q\gamma^\mu A_\mu\psi + m\psi$. This yields

$$\Box_E\psi + m^2\psi + \frac{i}{2}q\gamma^\mu\gamma^\nu F_{\mu\nu}\psi + \gamma^\mu(\nabla_\mu\gamma^\nu)(D_\nu\psi) = 0, \quad [37]$$

where we have defined the D'Alambertian operator in the presence of an electromagnetic field by $\Box_E = (\nabla_\mu + iqA_\mu)(\nabla^\mu + iqA^\mu)$. Using the Weyl representation, which has been discussed in this section, it becomes

$$\begin{aligned} i\frac{m}{\sqrt{n_{\dot{\nu}}}}\left[-\frac{\omega}{m}\nabla_0 n_{\dot{\nu}} + \nabla_\mu(n_{\dot{\nu}}v^\mu) + \frac{\omega}{m}\Box t\right] &+ \\ \sqrt{n_{\dot{\nu}}}\left[m^2v_\mu v^\mu + 2m\omega v^0 + \frac{\omega^2}{N^2} + m^2 - \frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}\right] &+ \\ (2NN^kF_{0k} + i\epsilon^{lj}{}_k\hat{F}_{lj}\tilde{\sigma}^k)R_R &+ \\ -(\hat{\nabla}_a\hat{e}^\alpha_b)\bar{\mathbb{S}}^a\mathbb{S}^b((mv_\alpha - \omega\delta^0_\alpha)R_R + D_\alpha R_R) &= 0. \end{aligned} \quad [38]$$

4. Energy Balance

From equation (38), we can identify the different energy contributions to the Fermi gas, and obtain an energy balance equation for fermions analogous to the one obtained for bosons in [1, 2]. In order to simplify the notations, we can re-write equation (38) in terms of the $\dot{\nu}$ coefficients with the understanding that the subindex R refers to each component $R = \dot{1}, \dot{2}$ individually. We get

$$\begin{aligned} i\left[-\omega\nabla_0 \ln(n_{\dot{\nu}}) + \frac{m\nabla_\mu(n_{\dot{\nu}} v^\mu)}{n_{\dot{\nu}}} + \frac{\omega}{n_{\dot{\nu}}}\Box t\right] &+ \\ 2m^2\left(K + \frac{1}{m}\omega v^0 + \frac{1}{2}U^N + U^Q\right) + E + U^S &= 0. \end{aligned} \quad [39]$$

This equation is valid for right handed fermions. The result is the same for left handed fermions changing $R_R \longrightarrow R_L$ in the first line, and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ in the second line.
The first line in eq. (39) describes the free density evolution of the fermions, while the contribution of the different energy terms appears in the second line. The first one is the kinetic energy $K_{\dot{\nu}}$ defined as

$$K = \frac{1}{2}v_\mu v^\mu. \quad [40]$$

The lapse potential U^N is given by

$$U^N = \frac{\omega^2}{m^2}\frac{1}{N^2} + 1. \quad [41]$$

It represents the energy contribution due to the chosen lapse function N. The quantum potential U^Q is defined as

$$U^Q = -\frac{1}{2m^2}\frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}. \quad [42]$$

The contribution of the electromagnetic interaction E is given by

$$\begin{aligned} E &= (2NN^k F_{0k} + i\epsilon^{lj}{}_k \hat{F}_{lj}\tilde{\sigma}^k), \quad [43] \\ &= 2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3) \\ &- e_{1\dot{\nu}}F_{13}\sqrt{\frac{n_{\ddot{\nu}}}{n_{\dot{\nu}}}} + i\left(e_{1\dot{\nu}}F_{12} + F_{23}\sqrt{\frac{n_{\ddot{\nu}}}{n_{\dot{\nu}}}}\right). \end{aligned}$$

It depends on the Faraday tensor, shift vector and lapse function that are related to the Pauli matrices. This relationship is due to the interaction between the electromagnetic field and the fermionic spin. Finally, the potential $U^S_{\dot{\nu}}$ describes the interaction between the spin and the geometry of space-time. It is given by

$$U^S = -\left((m\hat{v}_{Rd} - \omega_{\dot{\nu}}\hat{\delta}^0_d) + \frac{\hat{D}_\alpha\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}\right)\Gamma^d_{ba}\bar{\mathbb{S}}^a\mathbb{S}^b, \quad [44]$$

for $\dot{\nu} = \dot{1}, \dot{2}$, and by making the substitution $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ for $\dot{\nu} = \dot{3}, \dot{4}$. In the foregoing equations, the notation $|_{\dot{\nu}}$ means that we have to evaluate the quantity at the corresponding $\dot{\nu}$. Note that $U^S_{\dot{\nu}}$ disappears if we assume a flat space-time or if we consider particles without spin. Furthermore, $U^S_{\dot{\nu}}$ is constructed with the generalized gamma matrices (25), which are related to the spin (the Pauli matrices) and to the space-time geometry (tetrads).

5. Conclusions

A non-standard representation for fermions was worked using an analogy as in the boson and quantum mechanics case, where it was proposed the Madelung transformation. We extended this transformation for the spinor case, either Dirac or Weyl fermions. Thus, it was possible to get a successful hydrodynamic representation for fermions in an arbitrary framework coupled to an electromagnetic field. Although, the full equations that describe the Fermi gas behaviour are more complicated than in standard description. This is closer to the De Broglie-Bohm interpretation in quantum theory, where the measure problem can be solved by a statistic way. Furthermore, a non-obvious result using this new description was the first law of the thermodynamics or the energy balance equation, where different energy contributions of these kind of particles were found.
The main difference between the hydrodynamic representation of bosons [1, 2] and fermions, concerns the form of the Bernoulli equation. For bosons, after doing the Madelung transformation, we can separate the KG equation into real and imaginary parts. By contrast, for fermion particles we have to work with the complete equations of motion because the real and imaginary parts cannot be easily separated. This is related to the fact that the gamma matrices are a representation of the SO(1, 3) group and the generalized Madelung transformation used, because it only admits complex parameter to fulfill the Lorentz invariance.
The spin is a fundamental outcome of the Dirac equation [22], which combines both elements of special relativity and quantum mechanics, that was introduced to solve the problem of negative probability present in the KG equation – first proposed as a relativistic generalization of the Schrödinger equation. Here, we observe that the general relativistic Dirac equation involves an additional contribution due to geometry and spin through the generalized gamma and Pauli matrices. These terms arise from endowing a quantum field with a curvature (geometry) given by a metric in General Relativity. Such a contribution is absent in a flat space-time and in a system without spin as for a scalar field.
With this work we open the possibility of studying in detail the behavior of fermions in different situations (such as massive stars or dark matter halos harboring a central black hole), where general relativity effects may be important. We solved the problem of energy balance for both bosons and fermions. In this manner, we can compare the result of the hydrodynamic representation for classical and quantum fluids in the various geometries mentioned above.

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