Chaos and Complexity for Inverted Harmonic Oscillators

Le-Chen Qu[1,2], Jing Chen[1,2], Yu-Xiao Liu[1,2]

[1]Lanzhou Center for Theoretical Physics, Key Laboratory of Theoretical Physics of Gansu Province, School of Physical Science and Technology, Lanzhou University, Lanzhou 730000, China

[2]Institute of Theoretical Physics & Research Center of Gravitation, Lanzhou University, Lanzhou 730000, China

Introduction

The precise definition of quantum chaos is still an open question. Unlike classical chaos, one can only describe different aspects of quantum chaos [1, 2]. The precise definition of quantum chaos will help the understanding of thermalization, transport in quantum many-body systems and black hole information loss. Therefore, how to accurately define quantum chaos is indeed an important issue [1, 2].

The reason for the appearance of classical chaos is that the evolution of the system is very sensitive to the initial conditions due to the highly non-linearity of the equation of motion. And the distance between two adjacent points in the phase space increases as $e^{\lambda_L t}$, where λ_L is the Lyapunov exponent [3]. However, since Schrödinger equation is a linear equation, the evolution of a quantum system is not highly sensitive to the initial state in principle. This forces us to develop new chaotic probes for quantum systems.

Recent works have shown that circuit complexity can indeed detect quantum chaos [4, 5, 6]. In particular, it was found that chaotic behavior can be characterized by the complexity of bidirectional evolution. That is, first evolve a reference state forward with a Hamiltonian $\hat{H}$ and then evolve backward with a slightly different Hamiltonian $\hat{H}+\delta\hat{H}$. Finally, examine the complexity between the resulting state and the chosen reference state.

However, because the expression of complexity is usually very complicated, only the numerical fitting of Lyapunov exponent and scrambling time was obtained in Ref. [4]. In this work, by considering infinitesimal perturbation, we find the analytical expression of the circuit complexity and derive the Lyapunov exponent and scrambling time. As the most natural concept to measure the distance between states, the Loschmidt echo was proposed in Ref. [7] to describe quantum chaos. Similar to complexity, Loschmidt echo is defined as the inner product under bidirectional evolution. We also find that the Lyapunov exponent can indeed be obtained through the Loschmidt echo, which is consistent with Ref. [7].

Main results

We start with the Hamiltonian of the inverted harmonic oscillator, namely

$$\hat{H}=\frac{\hat{p}^2}{2m}+\hat{V},\quad \hat{V}=-\frac{1}{2}m\omega^2\hat{q}^2, \tag{1}$$

where m and ω are the mass and frequency of the inverted harmonic oscillator, respectively. Due to the simplicity of Gaussian states ψ, one can show that they are completely characterized by the so-called covariance matrix, the two-point functions defined by,

$$G^{ab}=\psi(\hat{\xi}^a\hat{\xi}^b+\hat{\xi}^b\hat{\xi}^a)\psi, \tag{2}$$

where the vector operator $\hat{\xi}^a=\left(\hat{q}g,\frac{\hat{p}}{g}\right)$. Note that g is a new gate scale, and its dimension is one. Considering the unitary transformation between Gaussian states, namely

$$\psi'=\hat{U}\left|\psi\right\rangle,\quad \hat{U}=e^{-\frac{1}{2}k_{ab}\hat{\xi}^a\hat{\xi}^b}, \tag{3}$$

one can rewrite it as a unique transformation of their corresponding covariance matrices as follows [8]

$$G'^{ab}=\psi'(\hat{\xi}^a\hat{\xi}^b+\hat{\xi}^b\hat{\xi}^a)\psi'=M^a{}_cG^{cd}M^b{}_d, \tag{4}$$

where

$$M=e^K,\quad K^a{}_b=\Omega^{ac}k_{cb},\quad \Omega^{ab}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{5}$$

As a consequence, we should expect that the circuit complexity of Gaussian states can be rewritten as a function of the covariance matrix. Particularly, it was found in [9] that the information about the circuit complexity is encoded in the relative covariance matrix that is defined by

$$\Delta=G_TG_R^{-1}. \tag{6}$$

where G_R and G_T denote the covariance matrix of the reference state and the target state, respectively. In the following, we denote the eigenvalue of greater than or equal to 1 as ρ. For Gaussian states, it was explicitly shown in Ref. [9] that the circuit complexity related to the F_2 cost function is given by

$$\mathcal{C}\left(G_R,G_T\right)=\frac{1}{2\sqrt{2}}\sqrt{\mathrm{Tr}\left[(\log\Delta)^2\right]}=\frac{1}{2}\log\rho. \tag{7}$$

For the Gaussian states, the inner product also depends on the relative covariance matrix,

$$\mathcal{I}=\left|\langle G_R\mid G_T\rangle\right|^2=\det\frac{\sqrt{2}\Delta^{1/4}}{\sqrt{\mathbb{1}+\Delta}}=\frac{2\sqrt{\rho}}{1+\rho}. \tag{8}$$

We choose the reference state as

$$\psi_R(q)=\left(\frac{am\omega}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}am\omega q^2\right), \tag{9}$$

where a is a dimensionless parameter that characterizes the Gaussian states. We consider a particular target state ψ_T by a backward and forward time evolution from the reference state,

$$\left|\psi_T\right\rangle=e^{i\hat{H}'t}e^{-i\hat{H}t}\left|\psi_R\right\rangle. \tag{10}$$

The perturbed Hamiltonian is defined by

$$\hat{H}'=\frac{\hat{p}^2}{2m}-\frac{1}{2}m(\omega+\delta\omega)^2\hat{q}^2\quad\text{with}\quad\frac{\delta\omega}{\omega}\ll 1. \tag{11}$$

In order to obtain a simple analytical expression of the chaotic behaviors, we focus on the infinitesimal perturbation by taking $\frac{\delta\omega}{\omega}\to 0$ to simplify the precise expression of ρ. Finally, we find that the leading terms of ρ are derived as

$$\begin{aligned}\rho\approx\rho_0=1&+\frac{\delta\omega^2}{\omega^2}\left(\frac{\left(1+a^2\right)^2\cosh(4\omega t)}{16a^2}-\cosh(2\omega t)\right)\\&+\frac{\delta\omega}{\omega}\left(\frac{\left(1+a^2\right)^2\cosh(4\omega t)}{8a^2}-2\cosh(2\omega t)\right.\\&\left.-\frac{\left(1+a^2\right)^2\delta\omega^2\cosh(6\omega t)}{16a^2\omega^2}+\frac{\left(1+a^2\right)^4\delta\omega^2\cosh(8\omega t)}{512a^4\omega^2}\right)^{1/2}\end{aligned} \tag{12}$$

where we still keep higher-order terms $\delta\omega^2,\delta\omega^4$ due to the competing factors $\cosh(\#\omega t)$ with exponentially increases in time. Note that there are four modes of exponential change over time, namely $\omega, 2\omega, 3\omega, 4\omega$.

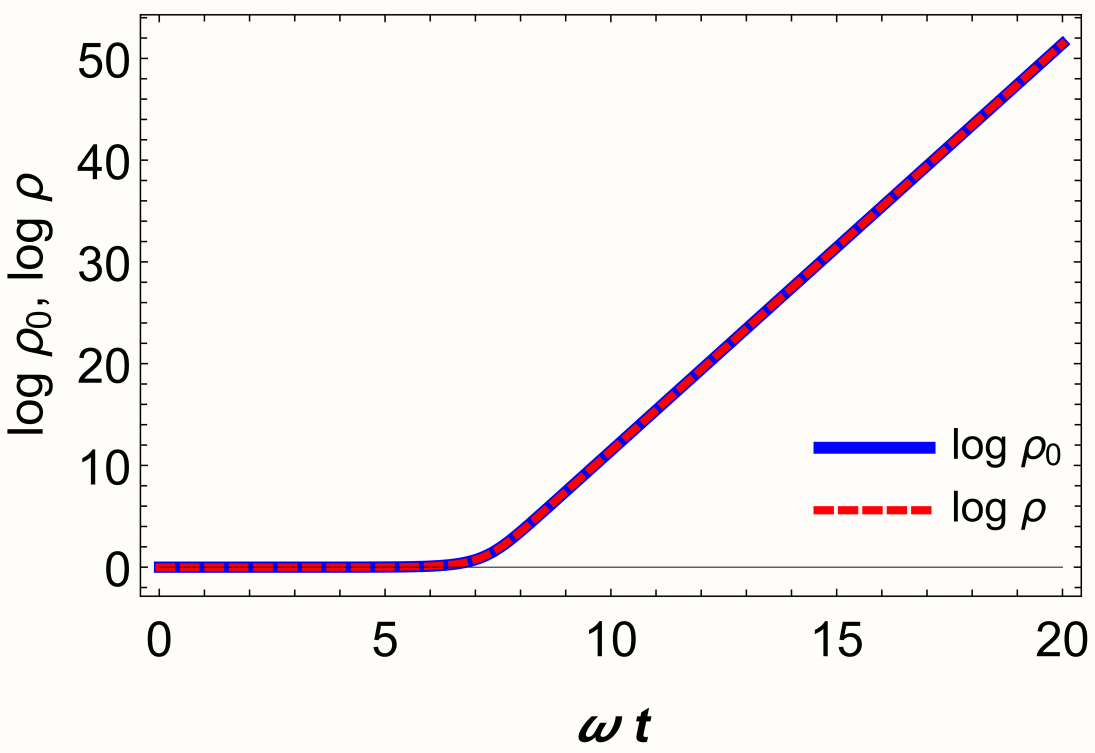

Figure 1: $\log\rho_0$ and $\log\rho$ vs time. The parameters are set as $a=2$ and $\delta\omega/\omega=10^{-6}$.

Summary and conclusions

In this work, we obtained the analytical expressions of the circuit complexity and Loschmidt echo. We found that the leading contributions of the circuit complexity $\mathcal{C}$ and Loschmidt echo $\mathcal{I}$ exhibit similar characteristics at late times,

$$\begin{aligned}\mathcal{C}(t)&\approx 2\omega\left(t-\frac{1}{2\omega}\log\frac{4a\omega}{(1+a^2)\delta\omega}\right),\\-\log\mathcal{I}(t)&\approx 2\omega\left(t-\frac{1}{2\omega}\log\frac{8a\omega}{(1+a^2)\delta\omega}\right).\end{aligned} \tag{13}$$

Through the above analytical results, we can read the Lyapunov exponents $\lambda_L^{\mathcal{C}}$ and $\lambda_L^{\mathcal{I}}$ and the scrambling times $t_s^{\mathcal{C}}$ and $t_s^{\mathcal{I}}$. The Lyapunov exponent $\lambda_L^{\mathcal{C}}$ is consistent with the result in [4] that was derived by using numerical fitting. Furthermore, we found that the scrambling time will diverge as the perturbation approaches zero because it takes infinite time for the system to respond to infinitesimal perturbations. We also found that the scrambling time explicitly depends on the choice of the reference states.

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