On energy for accelerating observers in black hole spacetimes

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Result

In Ref. [1] an energy for constantly accelerating observers (CAOs) in three black hole spacetimes, Bañados, Teitelboim, and Zanelli (BTZ), Schwarzschild and Schwarzschild-de Sitter, is derived. The expressions are in terms of acceleration, cosmological constant, and area, quantities measurable by the observers. Based on results from quantum fields in curved spacetime for the redshifted Hawking temperature, quasi-local entropy and thermodynamic-like laws may be explored.

The energy for CAOs

Although no notion of local energy exists in general spacetimes, there are useful notions of quasi-local energy for regions of spacetime [2, 3]. Defining energy is a matter of choice and, in particular, physical characterization: *of what* and *for whom* are we defining energy. Considerations in that characterization are as often practical as foundational. This work defines an energy for constantly accelerating observers (CAOs) in black hole spacetimes with a timelike Killing vector field.

The spacetimes considered here all have a timelike Killing vector field, ξ^a. Due to Killing's equation and the geodesic motion of a particle in free fall with momentum p^a, the energy $\mathcal{E} = -p_a \xi^a$ is conserved along the particle's history. A local observer with 4-velocity u^a measures the energy of this particle to be $E = -p_a u^a$ anywhere in the spacetime. This expression captures relevant aspects of the spacetime when the observers' 4-velocity is proportional to the Killing field ξ^a, $u^a = \xi^a/V$ with the 'redshift factor' $V = \sqrt{-\xi^a \xi_a}$. The change in the locally measured energy dE due to the flow of energy-momentum by the observers' surface is proportional to the conserved energy $d\mathcal{E}$,

$$dE = -p_a u^a = -p_a \frac{\xi^a}{V} = \frac{d\mathcal{E}}{V}. \tag{1}$$

One can arrive at this simple form of energy $dE = dM/V$ by considering the conserved flux $\delta T^a_b \xi^b$ of test matter fields with stress-energy tensor δT^{ab}, as shown in [1].

Sketch of the derivation

In the Schwarzschild metric the norm of the timelike Killing vector field gives $V = \sqrt{-\xi^a \xi_a} = \sqrt{1 - 2M/r}$. In the definition of energy one considers a one parameter family of solutions based on the mass M. For constantly accelerating observers as the asymptotic mass of the spacetime changes, the radial location of the observers changes as well. It is useful to consider variable mass m as a function of radius. The energy (1) becomes

$$E_g = \int \frac{dm|_g}{V} = \int_0^{r_g(M)} \left(1 - \frac{2m}{r}\right)^{-1/2} \left(\frac{m}{r}\right) \left(\frac{2r - 3m}{r - m}\right) dr, \tag{2}$$

which integrates to

Schwarzschild

$$E_g = g\frac{A}{4\pi} - \frac{1}{2g}\left[\ln\left(\sqrt{1+(gr)^2} - gr\right) + gr\sqrt{1+(gr)^2}\right]. \tag{3}$$

The energy E_g for CAOs may be expressed in terms of the areal radius and the acceleration.

Extensions

The non-rotating Bañados, Teitelboim, and Zanelli black hole solution in $(2+1)$-dimensional gravity [4] has energy

$$E_g = \int_0^r dE_g = \left(\frac{g}{2\pi}\right)\left(\frac{2\pi r}{4}\right)\left(1 - 1/(g\ell)^2\right), \tag{4}$$

Kottler [5] or Schwarzschild-de Sitter (SdS) spacetimes have energy

$$E_g = g\frac{A}{4\pi} + \frac{1}{2g}\left[\frac{\ln\left(gr\sqrt{1-\Lambda/g^2} + \beta\right)}{\sqrt{1-\Lambda/g^2}} - gr\beta\right], \tag{5}$$

with

$$\beta = \sqrt{1 + (gr)^2(1 - \Lambda/g^2)}.$$

By analogy one might expect that these definitions should result in analogs of black hole thermodynamics. Although there are indications that in the case of the non-rotating BTZ and Schwarzschild spacetimes that these may hold, further study is required [1].

References

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