A summary of: Constraining modified gravity with quantum optomechanics

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In our article [1], we derive the best possible bounds that can be placed on Yukawa– and chameleon–like modifications to the Newtonian gravitational potential with a cavity optomechanical quantum sensor. By modelling the effects on an oscillating source-sphere on the optomechanical system from first-principles, we derive the fundamental sensitivity with which these modifications can be detected in the absence of environmental noise. In particular, we take into account the large size of the optomechanical probe compared with the range of the fifth forces that we wish to probe and quantify the resulting screening effect when both the source and probe are spherical. Our results show that optomechanical systems in high vacuum could, in principle, further constrain the parameters of chameleon-like modifications to Newtonian gravity.

I. INTRODUCTION

General Relativity (GR) is one of the most successful theories of nature, but there are compelling reasons to explore modifications to the behaviour of gravity on both large and small scales. Most of the precise predictions of GR have consistently been demonstrated experimentally. While a natural part of GR, a cosmological constant poses a theoretical challenge to particle physics since the small observed value is inherently sensitive to high-energies, requiring delicate balancing [2]. Furthermore, many theories of high energy physics that attempt to solve this and other problems – such as building a consistent quantum theory of gravity – predict deviations from GR. These theories are collectively known as *modified gravity theories*.

Modified gravity theories, however, typically face a difficult challenge in the form of solar system tests of Newton's laws. Models that differ from GR significantly enough to explain the observed acceleration of the Universe on large scales are typically ruled out by their predicted deviations on smaller scales (solar system and laboratory tests) [3–5]. An approach considered by many authors is the chameleon mechanism [6–8]; the basic idea is to add a scalar field that couples directly to gravity in a manner that depends on the local density of matter. In high-density regions, such as inside a galaxy, the effects of modified gravity are screened out, allowing the theory to evade solar system tests. In the low-density void regions between galaxies, however, the effects of modified gravity would be unscreened.

If such a density-dependent gravity mechanism is at play, it ought to be detectable in principle by high-precision laboratory experiments. In particular, the fundamental sensitivity improvements offered by quantum systems are especially promising [9]. At the moment, the detection of modified gravity, and in particular, chameleon fields, has been explored through a diverse variety of methods [10].

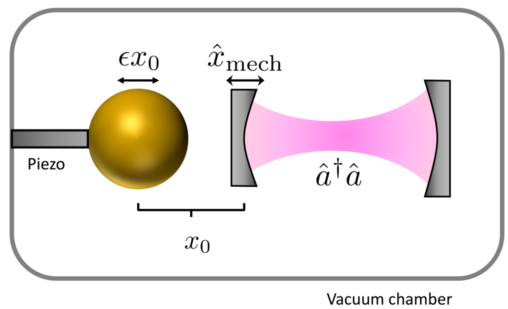

FIG. 1. A gold source mass attached to a shear piezo oscillates to create a time-varying gravitational field. The field, which potentially contains deviations from Newtonian gravity, is detected by an optomechanical probe system where the photon number $\hat{a}^\dagger \hat{a}$ couples to the mechanical position $\hat{x}_{\mathrm{mech}}$ as $\hat{a}^\dagger \hat{a} \hat{x}_{\mathrm{mech}}$, here presented as a moving-end mirror in a Fabry–Pérot cavity. The amplitude ϵx_0 of the source mass oscillation is a fraction of the total distance x_0 between the systems. By accounting for the vacuum background density, we may also compute bounds on the parameters of the chameleon screening mechanism.

An additional approach to detecting the small-scale effects of modified gravity and screening is to take advantage of recent developments in the field of optomechanics, where a small mechanical element is coupled to a laser through radiation-pressure [11, 12] (see figure 1). The key question we seek to answer in this work is: what fundamental range of parameters of modified gravity theories could ideally be excluded with a quantum optomechanical sensor? To address this question, we consider an idealised system described by a nonlinear, dispersive, optomechanical Hamiltonian which couples the optical and mechanical degrees of freedom through a nonlinear radiation-pressure term given by (in the absence of an external gravitational field):

$$\hat{H}_0 = \hbar\,\omega_{\mathrm{c}}\,\hat{N}_a + \hbar\,\omega_{\mathrm{mech}}\,\hat{N}_b - \hbar\,k(t)\,\hat{N}_a\,\big(\hat{b}^\dagger + \hat{b}\big), \quad (1)$$

where ω_{c} and ω_{mech} are the oscillation frequencies of the optical cavity mode and mechanical mode respectively, with annihilation and creation operators $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$. We have also defined $\hat{N}_a = \hat{a}^\dagger\hat{a}$ and $\hat{N}_b = \hat{b}^\dagger\hat{b}$ as the photon and phonon number operators.

The coupling $k(t)$ is the (potentially time-dependent) characteristic single-photon interaction strength between the number of photons and the position of the mechanical element. It takes on different forms depending on the optomechanical platform in question. Modulation of the optomechanical coupling can be introduced in different ways depending on the experimental platform in question. For example, the mechanical frequency of a cantilever can be modified by applying an oscillating electric field [13, 14], and a modulated coupling arises naturally through the micro-motion of a levitated system in a hybrid electro-optical trap [15–17].

The Hamiltonian (1) is often linearised for a strong coherent input drive, however the fully nonlinear (in the sense of the equations of motion) Hamiltonian is a more fundamental description. While all quantum systems are affected by noise, we here assume that the coherence times can be made long enough for the measurement protocol to be carried out. As a result, our analysis explores the bounds in the absence of environmental noise and decoherence. We then consider the gravitational field that arises when a source mass is placed next to the sensor as shown in figure 1. Since it is often difficult in experiments to distinguish a signal against a constant noise floor, we consider an oscillating source mass, which gives rise to a time-dependent gravitational field.

When treating the system in a closed and ideal setting, we can model the initial state as a separable state of the light and the mechanical element. For the optical state, we consider injecting squeezed light into the cavity. Squeezed light has been shown to fundamentally enhance the sensitivity to displacements [9]. By including squeezing here, we generalise our scheme to include these input states. The state of the mechanical element, on the other hand, is most accurately described as thermal at a non-zero temperature. With these assumptions, the initial state of the system can be written as

$$\hat{\varrho}(0) = |\zeta\rangle\langle\zeta| \otimes \sum_{n=0}^{\infty} \frac{\tanh^{2n} r_T}{\cosh^2 r_T}\,|n\rangle\langle n|\ , \quad (2)$$

where $|\zeta\rangle = \hat{S}_\zeta\,|\mu_{\mathrm{c}}\rangle$ is a squeezed coherent state of the optical field where $\hat{S}_\zeta = \exp\big[(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2})/2\big]$ and where the coherent state satisfies $\hat{a}\,|\mu_{\mathrm{c}}\rangle = \mu_{\mathrm{c}}\,|\mu_{\mathrm{c}}\rangle$. The squeezing parameter can also be in spherical polar form as $\zeta = r_{\mathrm{sq}}\,e^{i\varphi}$. The parameter r_T of the thermal state arises from the Bose–Einstein distribution and is defined by $\tanh r_T = \exp\Big[-\frac{\hbar\,\omega_{\mathrm{mech}}}{2\,k_{\mathrm{B}}\,T}\Big]$, where T is the temperature of the system and k_{B} is Boltzmann's constant.

II. CHAMELEON FORCE AND YUKAWA POTENTIAL

To determine whether our analysis is valid in the case of a chameleon field, we derive the time-dependent potential that results from the source mass from first principles, where we find that a potential that moves with the mass is the correct choice for non-relativistic velocities. Another key consideration for optomechanical systems is the relatively large size of the optomechanical probe which contributes significantly to the chameleon screening of the fifth force in the envisioned setup of the quantum experiment we consider here (as opposed to, for example, cold atoms, where the screening length of the atomic probes is very small). To take the finite screening length into account, we go beyond the common approximation that the probe radius is small compared to the range of the chameleon field and derive analytic expressions for the modified force seen by the probe.

The net effect of the chameleon scalar field ϕ is to modify the effective Newtonian potential affecting a test particle. In this work, we consider a chameleon model with an effective self-interaction potential

$$V_{\mathrm{eff}}(\phi) = \frac{\Lambda^{4+n}}{\phi^n} + \frac{\phi\rho}{M}(\hbar c)^3. \quad (3)$$

We explore only the case $n = 1$ in this work: other models and choices of n are possible, but we choose this specific example to demonstrate how the method works in principle. This model has two parameters; Λ, which characterises the energy scale of the chameleon's self-interaction potential; and M (here chosen to be a mass to give the correct units for a potential), which determines how strongly the chameleon field affects test particles and arises from the non-minimal coupling of the chameleon field to curvature.

For $n = 1$ the background value of the field, ϕ_{bg}, in an environment of constant mass density ρ_{bg} is given by

$$\phi_{\mathrm{bg}} = \sqrt{\frac{M\Lambda^5}{\rho_{\mathrm{bg}}(\hbar c)^3}}. \quad (4)$$

In the centre of the source, the chameleon field reaches its minimum value of ϕ_S. The mass of the chameleon field, m_{bg}, is density dependent and given by

$$m_{\mathrm{bg}}c^2 = \left(\frac{4\,\rho_{\mathrm{bg}}^3(\hbar c)^9}{M^3\Lambda^5}\right)^{1/4}. \quad (5)$$

To obtain an expression for the chameleon field around a spherical matter distribution, we use the same asymptotic matching approach as Burrage *et al.* [18]. In the limit where the probe radius is much smaller than the distance between the probe and the source sphere $R_P \ll |\mathbf{X}_S(t)|$, we find the following expression for the force:

$$F = -\frac{G\,M_S m}{|\mathbf{X}_S(t)|^2}\left[1 + \alpha_{\mathrm{bg},P}\left(1 + \frac{|\mathbf{X}_S(t)|}{\lambda_{\mathrm{bg}}}\right) e^{-|\mathbf{X}_S(t)|/\lambda_{\mathrm{bg}}} \times \right.$$
$$\left. \times\, f(R_P/\lambda_{\mathrm{bg}}, |\mathbf{X}_S(t)|/\lambda_{\mathrm{bg}})\right], \tag{6}$$

where M_S and m are the mass of the source sphere and the probe sphere, respectively, $\lambda_{\mathrm{bg}} = \frac{\hbar}{m_{\mathrm{bg}} c}$ and the sensor-dependent fifth-force strength is defined as

$$\alpha_{\mathrm{bg},P} = \frac{2M_{\mathrm{P}}^2}{M^2}\,\xi_S\,\xi_P\,, \tag{7}$$

where we have added the subscript 'P' to denote that screening from the probe is here taken into account, $M_{\mathrm{P}} \approx \sqrt{\hbar c/(8\pi G)} = 4.341 \times 10^{-9}$ kg is the Planck mass. Furthermore, ξ_S and ξ_P (labelled S for the source and P probe, respectively), are given by

$$\xi_{S,P} = \begin{cases} 1, & \rho_{S,P} R_{S,P}^2 < 3M\,\phi_{\mathrm{bg}}/(\hbar c), \\ 1 - \frac{S_{S,P}^3}{R_{S,P}^3}, & \rho_{S,P} R_{S,P}^2 > 3M\phi_{\mathrm{bg}}/(\hbar c)\,. \end{cases} \tag{8}$$

where $\rho_{S,P}$ and $R_{S,P}$ are the density and the radius of the source/probe, respectively, and for the Chameleon model we consider, S_i with $i = S, P$ is found by solving a cubic equation. In the $m_{\mathrm{bg}} R_i c/\hbar \to 0$ and $\phi_{\mathrm{bg}} \gg \phi_i$ limits, this reduces to

$$S_i = R_i\sqrt{1 - \frac{8\pi M}{3M_i}\frac{R_i\phi_{\mathrm{bg}}}{\hbar c}}, \tag{9}$$

which is the result found by Burrage *et al.* [18]. S_i parametrises the screening effect of the chameleon mechanism for a spherical source/probe: for example, when S_i is much lower than R_i, the field is effectively unscreened while for $S_i \approx R_i$ the field is heavily screened. Finally, the function f is a form-factor which approaches 1 in the $x = m_{\mathrm{bg}} R_P c/\hbar = R_P/\lambda_{\mathrm{bg}} \to 0$ limit, in which case equation (6) reduces to the result of Burrage *et al.* [18] for the force between two spheres. Since spherical probes or source masses generally maximise the screening [19], equation (6) can be interpreted as a conservative estimate of the screening due to the shielding from the probe.

In order to compute the sensitivity of the optomechanical system, we need to include the force on the sensor shown in equation (6) into the dynamics of the optomechanical system. We start by assuming that the source mass and the mechanical element of the optomechanical system are constrained to move along the x-axis. The full optomechanical Hamiltonian including the modified gravitational potential can then be written as

$$\hat{H}(t) = \hat{H}_0 - \mathcal{G}_{\mathrm{Cha}}(t)\,x_{\mathrm{zpf}}\left(\hat{b}^\dagger + \hat{b}\right), \tag{10}$$

where $\hat{H}_0$ is given in equation (1) and where the time-dependent modified Newtonian gravitational force is contained in the second term.

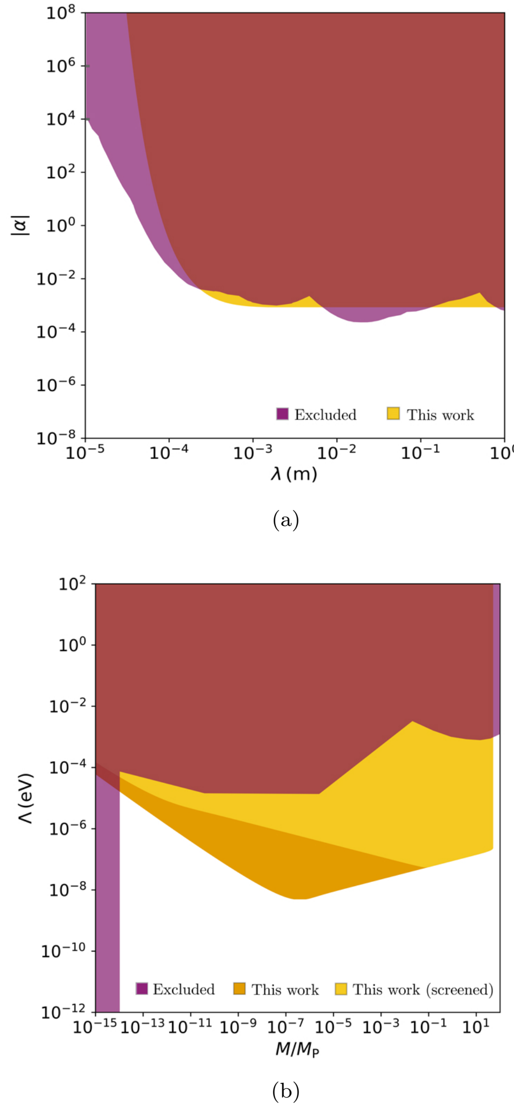

FIG. 2. Comparison between predictions (this work) and known experimental bounds (pink region). Both plots show the convex hull (yellow) of the bounds derived in this work. Plot (a) shows the bounds in terms of the Yukawa parameters α and λ, while Plot (b) shows the bounds in terms of the chameleon screening parameters M and Λ. Plot (b) also includes the bounds (yellow) for when the optomechanical probe contributes to the screening of the chameleon field. The pink areas represent the experimentally excluded regions based on figure 8 of [20] and recent results presented in [21] (see figure 6). The experimentally excluded regions in (b) are based on those reported in Ref [10].

It is possible to obtain the solution numerically, but in order to obtain analytic expressions, we choose to linearise the modification of the force for small oscillations of the source-mass. We let the time-dependent distance between the systems $x_S(t)$ be given by:

$$x_S(t) = x_0\,\left(1 - \epsilon\cos(\omega_0\,t + \phi_0)\right), \tag{11}$$

where ϵ is a dimensionless oscillation amplitude defined as a fraction of x_0, where ω_0 is the oscillation frequency and ϕ_0 is a phase shift that we specify later in order to maximize the sensitivity. We obtain

$$\mathcal{G}_{\rm Cha}(t) \approx -\frac{GM_S m}{x_S^2(t)} - m g_{\rm N} \left(\kappa + \sigma \epsilon \cos(\omega_0 \, t + \phi_0)\right) . \tag{12}$$

For the parameter regimes considered in this work, we find that κ and σ are

$$\kappa = \alpha_{{\rm bg},P} \, e^{-x_0/\lambda_{\rm bg}} \left(1 + \frac{x_0}{\lambda_{\rm bg}}\right), \\ \sigma = \alpha_{{\rm bg},P} \, e^{-x_0/\lambda_{\rm bg}} \left(2 + 2\frac{x_0}{\lambda_{\rm bg}} + \frac{x_0^2}{\lambda_{\rm bg}^2}\right) . \tag{13}$$

If the screening of both source and probe can be neglected, we obtain the limit $\alpha_{{\rm bg},P} \to \alpha$ and $\lambda_{\rm bg} \to \lambda$ and the effect of the Chameleon force can be associated with a conventional Yukawa potential

$$V(r) = -\frac{G\, M_S m}{r} \left(1 + \alpha \, e^{-r/\lambda}\right), \tag{14}$$

where α parametrises the intrinsic difference in strength between the Yukawa-like fifth force and gravity, while λ parametrises the range of this fifth-force.

III. FUNDAMENTAL SENSITIVITY BOUNDS

Using tools from quantum information theory and quantum metrology such as the Quantum Fisher Information (QFI), we are then able to estimate the fundamental sensitivity for detecting deviations from Newtonian gravity. The connection to sensitivity stems from the fact that the QFI provides a lower bound to the variance $\mathrm{Var}(\theta)$ of θ through the quantum Cramér–Rao bound [22, 23]:

$$\mathrm{Var}(\theta) \geq \frac{1}{\mathcal{M}\,\mathcal{I}_\theta}, \tag{15}$$

where $\mathcal{M}$ is the number of measurements or probes used in parallel. The standard deviation of θ is then given by $\Delta\theta = 1/\sqrt{\mathcal{M}\,\mathcal{I}_\theta}$. For unitary dynamics and mixed initial states written in the form of $\hat{\varrho} = \sum_n \lambda_n \, |\lambda_n\rangle\langle\lambda_n|$, the QFI can be written in dependence of the operator $\hat{\mathcal{H}}_\theta = -i\hat{U}_\theta^\dagger \partial_\theta \hat{U}_\theta$, where $\hat{U}_\theta$ is the unitary operator that encodes the parameter θ into the system. In our case, $\hat{U}(\theta)$ is the unitary operator that arises from the Hamiltonian in equation (10), and the effect we wish to estimate is the effect of the Chameleon force on the probe. Therefore, in order to compute $\mathcal{I}_\theta$, we must first solve the time-evolution of the system, which is often challenging when the signal is time-dependent, as is the case for us here. Some of these challenges can however be addressed by making use of a previously established method for solving the Schrödinger equation using a Lie algebra approach [24]. Details of this solution were first used to study a purely Newtonian time-dependent gravitational potential [25].

We obtain a compact expression for the QFI that represents the sensitivity with which modifications to Newtonian gravity can be detected. In our case, we let the parameter θ of interest be either κ or σ. By then applying the Cramér–Rao bound, we can derive the standard deviation for each parameter. We then consider the ratios $\Delta\kappa/\kappa$ or $\Delta\sigma/\sigma$, which describe the relative error of the collective measurements. In this work, we say that we can distinguish modifications to the Newtonian potential if the error in κ and σ is smaller than one, that is, when $\Delta\kappa/\kappa < 1$ or $\Delta\sigma/\sigma < 1$. Note that, to find the sensitivity to the actual values of, for example, α and λ, we would need a full multi-parameter likelihood analysis, which requires us to go beyond the regular error-propagation formula for the parameter we consider here. Such an analysis is currently beyond the scope of this work. Instead, we focus mainly on detecting σ, since it is the amplitude of the time-dependent signal.

IV. RESULTS AND CONCLUSIONS

For the case of constant optomechanical coupling, we find that the sensitivities $\Delta\kappa$ and $\Delta\sigma$ are given by

$$\Delta\kappa = \frac{1}{\sqrt{\mathcal{M}}\, g_{\rm N}} \frac{1}{\Delta\hat{N}_a} \sqrt{\frac{2\hbar\,\omega_{\rm mech}^5}{m}} \frac{1}{8\pi\, n\, k_0}, \tag{16}$$

$$\Delta\sigma = \frac{1}{\sqrt{\mathcal{M}}\, g_{\rm N}} \frac{1}{\Delta\hat{N}_a} \sqrt{\frac{2\hbar\,\omega_{\rm mech}^5}{m}} \frac{1}{4\pi\, n\, k_0\, \epsilon}, \tag{17}$$

where n is an integer, and for an optomechanical coupling $k(t) \equiv k_0$ and phase $\phi_0 = \pi$, and where the variance $(\Delta\hat{N}_a)^2$ of the photon number can be optimized by choosing $e^{-i\varphi/2}\mu_{\rm c}$ is completely imaginary as [25]

$$(\Delta\hat{N}_a)^2 = |\mu_{\rm c}|^2 e^{4r_{\rm sq}} + \frac{1}{2}\sinh^2(2\, r_{\rm sq}) \, . \tag{18}$$

When choosing a sinusoidal modulation with $k(t) = k_0 \cos(\omega_k \, t)$, where k_0 is the amplitude of the modulation and ω_k is the modulation frequency, we find that $\Delta\kappa$ is increase by a factor 2 and $\Delta\sigma$ is decreased by a factor $2/(\pi n)$.

Our main results include the bounds presented in figure 2, which shows the parameter ranges of modified gravity theories that could potentially be excluded with an ideal optomechanical sensor. The bounds are computed for a specific set of experimental parameters. To facilitate investigations into additional parameter regimes, we have made the code used to compute the bounds available (https://github.com/sqvarfort/modified-gravity-optomech). While experiments are unlikely to achieve the predicted sensitivities due to noise and systematic effects, our bounds constitute a fundamental limit for excluding effects beyond Newtonian gravity given the experimental parameters in question. Such effects are discussed in detail in the discussion section of our article together with forces that arise from the Casimir effect.

Our results show that optomechanical sensors could, in principle, be used to improve on existing experimental bounds for the chameleon screening mechanism, although more work is needed to evaluate the prospects for using experimental optomechanical systems as probes for modified gravity.

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