ON THE CHANDRASEKHAR LIMIT IN GENERALIZED UNCERTAINTY PRINCIPLES

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Based on: arXiv:2202.13904 [gr-qc] with Yen Chin Ong

Heisenberg:

$$\Delta x \Delta p \sim \frac{\hbar}{2},$$

GUP:

$$\Delta x \Delta p \sim \frac{\hbar}{2}[1 + \beta^2 (\Delta p)^2],$$

where $\beta = 1/(M_p c)$ with $M_p = \sqrt{\frac{\hbar c}{G}}$ being the Planck mass

GUP*:

$$\Delta x \Delta p \sim \frac{\hbar}{2}\left[-\beta \Delta p + \frac{1}{1 - \beta \Delta p}\right],$$

See: Won Sang Chung and Hassan Hassanabadi, EPJC 79 (2019) 213

Classical limits $\hbar \to 0$ of GUP and GUP* are different:

- For GUP: $\Delta x \sim G \Delta p/(2c^3)$
- For GUP*: $\Delta x \Delta p \sim 0$
- For Heisenberg: $\Delta x \Delta p \sim 0$

Physics of GUP*:

- Existence of a minimal length
- Existence of a maximum momentum $p = M_P c$
- Connection to Doubly Special Relativity

- Generalized Uncertainty Principles encode both gravitational and quantum mechanical effects
- Similarly, also Chandrasekhar mass depends both on G and on $\hbar$:

$$M_{\mathrm{Ch}} \sim \frac{1}{m_e^2}\left(\frac{\hbar c}{G}\right)^{3/2}$$

This is the mass limit for a white dwarf in hydrostatic equilibrium when gravity is described by General Relativity

Question:

Does the formulation of the uncertainty principle affect the mass limit of a white dwarf?

To show the existence of Chandrasekhar limit in GUP has not been a trivial task:

- JCAP 09 (2018) 015
- Phys. Rev. D 98 (2018) 126018
- Annals Phys. 374 (2016) 434
- R. Soc. Open Sci. 8 (2021) 210301

DEGENERATE FERMION GAS IN GUP*

Number density:

$$n = \frac{1}{\pi^2\hbar^3}\int_0^{p_F} p^2(1-\beta p)dp = \frac{\xi^3\alpha(4-3\tilde{\beta}\xi)}{12m_e c^2},$$

Pressure:

$$P = \frac{1}{\pi^2\hbar^3}\int_0^{p_F} p^2(1-\beta p)(E_F - E_p)dp\,,$$

where

$$E_F = \sqrt{(cp_F)^2 + (m_e c^2)^2}\,, \qquad E_p = \sqrt{(cp)^2 + (m_e c^2)^2}\,.$$

$$P = \frac{\alpha}{20}\left[\frac{5}{2}\ln(\sqrt{\xi^2+1}+\xi) - \left(\tilde{\beta}\xi^4 - \frac{4\tilde{\beta}\xi^2}{3} - \frac{5\xi^3}{3} + \frac{8\tilde{\beta}}{3} + \frac{5\xi}{2}\right)\sqrt{\xi^2+1} + \frac{8\tilde{\beta}}{3}\right].$$

Internal kinetic energy of the degenerate electron gas

$$\begin{aligned}\varepsilon &= \frac{1}{\pi^2\hbar^3}\int_0^{p_F} p^2(1-\beta p)(E_p - m_e c^2)dp \\ &= \left[\frac{\tilde{\beta}\xi^4}{4} - \frac{\xi^3}{3} - \frac{2\tilde{\beta}}{15} - \frac{\ln(\sqrt{\xi^2+1}+\xi)}{8} - \left(\frac{\tilde{\beta}}{5}\left(\xi^4 + \frac{\xi^2-2}{3}\right) - \frac{\xi}{4}\left(\xi^2 + \frac{1}{2}\right)\right)\sqrt{1+\xi^2}\right]\alpha\end{aligned}$$

Rest mass density: $\rho_0 = m_u \mu_e n(\xi)$
Energy density:

$$\tilde{\varepsilon}(\xi) = \rho_0(\xi)c^2 + \varepsilon(\xi)\,,$$

Relativistic adiabatic index:

$$\begin{aligned}\gamma &= \frac{\tilde{\varepsilon}+P}{P}\frac{dP}{d\tilde{\varepsilon}} \\ &= \frac{5\xi^5(3\tilde{\beta}\xi-4)^2}{\left[30\ln(\sqrt{\xi^2+1}+\xi) - 2\left(6\tilde{\beta}\xi^4 - 8\tilde{\beta}\xi^2 - 10\xi^3 + 16\tilde{\beta} + 15\xi\right)\sqrt{\xi^2+1} + 32\tilde{\beta}\right](1-\tilde{\beta}\xi)\sqrt{\xi^2+1}}\end{aligned}$$

p_F is the Fermi momentum
$\xi := p_F/(m_e c)$
$\tilde{\beta} := m_e c\beta$
$\alpha := m_e^4 c^5/(\hbar^3\pi^2)$
Note the phase space volume:
$dp \to (1-\beta p)dp$
$m_u = 1.66 \times 10^{-24}$ g
$\mu_e = A/Z$
A mass number
Z atomic number
$j := m_u\mu_e/m_e$.

$$\begin{aligned}P_{\xi\to\infty} &\approx \frac{\alpha\xi^4}{4}\left(\frac{1}{3} - \frac{\tilde{\beta}\xi}{5}\right) \\ \varepsilon_{\xi\to\infty} &\approx \frac{\alpha\xi^4(5-4\tilde{\beta}\xi)}{20} \\ \tilde{\varepsilon}_{\xi\to\infty} &\approx \frac{\alpha\xi^4}{4}\left[1 - \left(j + \frac{4\xi}{5}\right)\tilde{\beta}\right]\end{aligned}$$

TOLMAN-OPPENHEIMER-VOLKOFF HYDROSTATIC EQUILIBRIUM

$$\frac{dP}{dr} = -\frac{G(\tilde{\varepsilon}+P)(M+4\pi P r^3/c^2)}{c^2 r(r-2GM/c^2)},$$
$$\frac{dM}{dr} = \frac{4\pi\tilde{\varepsilon}r^2}{c^2}.$$

Introduce dimensionless radial coordinate $\eta = (\alpha/(m_e c^2))^{1/3} r$ and dimensionless white dwarf mass $v = M/(m_e \pi)$:

$$\frac{d\xi}{d\eta} = \frac{q\sqrt{\xi^2+1}(\sqrt{\xi^2+1}+j-1)[\tau\eta^3+5v]}{5\xi(2qv-\eta)\eta};$$
$$\tau = \frac{5}{2}\ln(\sqrt{\xi^2+1}+\xi) - \frac{(6\tilde{\beta}\xi^4-8\tilde{\beta}\xi^2-10\xi^3+16\tilde{\beta}+15\xi)\sqrt{\xi^2+1}}{6}+\frac{8\tilde{\beta}}{3},$$
$$\frac{dv}{d\eta} = -\eta^2\left[\frac{\ln(\sqrt{\xi^2+1}+\xi)}{2}+\left(\frac{4\tilde{\beta}\xi^4}{5}+\frac{4\tilde{\beta}\xi^2}{15}-\xi^3-\frac{8\tilde{\beta}}{15}-\frac{\xi}{2}\right)\sqrt{\xi^2+1}+(j-1)\left(\tilde{\beta}\xi-\frac{4}{3}\right)\xi^3+\frac{8\tilde{\beta}}{15}\right]$$

In the non-relativistic regime for which $\xi \approx 0$ and considering $v \approx 0$, the approximated version of the TOV equations are

$$\frac{d\xi}{d\eta} \approx -\frac{jqv}{\eta^2\xi}, \qquad \frac{dv}{d\eta} \approx \frac{4j\eta^2\xi^3}{3},$$

Try a solution of the type $\xi \sim \eta^s$, $v \sim \eta^t$:

$$s-1 = t-2-s, \quad t-1 = 2+3s, \qquad \Rightarrow \qquad v \sim \eta^{-3} \qquad \Rightarrow \qquad R \sim M^{-\frac{1}{3}}.$$

HIGH MOMENTUM REGIME

High momentum limit: for $\xi \to 1/\tilde{\beta}$ the TOV equation for the mass reduces to

$$\frac{dv}{d\eta} = \eta^2\theta_2\,, \qquad \theta_2 = \frac{(16\tilde{\beta}^4 + 7\tilde{\beta}^2 + 6)\sqrt{1+\tilde{\beta}^2} - 16\tilde{\beta}^5 + 10(j-1)\tilde{\beta} - 15\tilde{\beta}^4 \ln\frac{\sqrt{1+\tilde{\beta}^2}+1}{\tilde{\beta}}}{30\tilde{\beta}^4}\,,$$

from which the relation $v = \frac{\eta^3\theta_2}{3}$, e.g. $M \propto R^3$, is easily obtained.

Equation for the momentum:

$$\frac{d\xi}{d\eta} \approx \frac{(\xi+j)(4\xi^2-5)\eta}{30(s\eta^2-1)}\,,$$

which delivers the implicit solution

$$\frac{\ln|s\eta^2-1|}{60s} + \frac{1}{4j^2-5}\left(\ln\frac{\sqrt{4\xi^2-5}}{\xi+j} + \frac{\sqrt{5}j}{5}\ln\frac{2\sqrt{5}\xi+1}{2\sqrt{5}\xi-1}\right) + C = 0,$$

with C being an arbitrary constant of integration. Since $\tilde{\beta} \approx 0$ we have

$$\frac{\tilde{\beta}^2}{8}\ln\frac{2\eta^2}{15\tilde{\beta}^2} + \frac{5\ln(2)\xi - 4j}{5(4j^2-5)\xi} + C \approx 0\,,$$

which provides

$$\eta(\xi) = \frac{\sqrt{30}\tilde{\beta}}{2}\exp\left(\frac{4[4j - 20Cj^2\xi + 25C\xi - 5\ln(2)\xi]}{5\tilde{\beta}^2\xi(4j^2-5)}\right).$$

In the context of GUP, the Chandrasekhar limit was identified as $v_{\max} := \lim_{\xi\to\infty} v(\eta(\xi))$.

In the case of GUP*, we set instead $v_{\max} := \lim_{\tilde{\beta}\to 0} v(\eta(\xi = 1/\tilde{\beta}))$.

$$v_{\max} \to \frac{1}{15\tilde{\beta}^4}\left(\frac{\sqrt{30}\tilde{\beta}}{2}e^{-16C/\tilde{\beta}^2}\right)^3.$$

Choosing the integration constant to be $C \sim \frac{\tilde{\beta}^2}{16}\ln(\sqrt{15/2}\tilde{\beta}^{2/3})$, we get $v_{\max} \sim \frac{1}{\tilde{\beta}^3}$, and $M_{\max} \sim \frac{M_p^3}{m_e^2}$ consistent with Chandrasekhar limit. We also get $\eta(\xi) \to \frac{\sqrt{30}\tilde{\beta}}{2}e^{-\frac{\ln(2)}{\beta^2 j^2}}$ in the high-momentum limit, which is arbitrarily close to zero for $\tilde{\beta} \sim 0$.

For GUP* the Chandrasekhar limit arises when $\eta \to 0$, e.g. when the size of the white dwarf is vanishingly small.

PHYSICAL REMARKS:

- Classical limit of GUP is equivalent to $\Delta x \sim G\Delta p/2c^3$. Interpretation: in the classical limit but when gravitational effect is dominant, a particle of mass m cannot be localized to within an uncertainty Δx. In the language of Heisenberg's microscope, if one focuses a beam of light with small enough wavelength (high enough energy) in order to localize the particle, an event horizon is formed and so we cannot locate the particle to within the horizon scale $\sim Gm/c^2$. The change in the momentum of the particle is γmv, which for v close to c but still bounded away from c, is $O(1)mc$ (e.g., at $v = 0.9c$, $\gamma mv \approx 2mc$). Thus we see that $\Delta x \sim G\Delta p/2c^3$ does lead to $\Delta x \sim Gm/c^2$.

- GUP combines the effect of Heisenberg's uncertainty principle with that of the uncertainty due to horizon formation in the presence of gravity, therefore in the classical limit it still reduces to the classical effect involving gravity, and not to the flat spacetime limit (unless $G \to 0$). On the other hand, the classical limit of GUP* reduces to that of flat spacetime.

- From this perspective it is not surprising that the Chandrasekhar limit for GUP* corresponds to a zero radius star (just like when the original Heisenberg's uncertainty principle is used), as there is no cutoff that corresponds to horizon formation. We emphasize that this is not a shortcoming: as in the original Chandrasekhar limit, it is an asymptotic state that is not attainable, as complicated stellar physics often lead to instabilities that result in most *stable* white dwarfs being bounded away from the Chandrasekhar limit.

- This actually raises an interesting conceptual question for the generalized/modified uncertainty principle community. Often it is useful to have a guiding principle or criterion that one can rely on when attempting to study quantum gravity phenomenology. Should we require that the classical limit recover the flat spacetime limit or the horizon formation criterion? Should a quantum uncertainty be put on equal footing with a classical one?