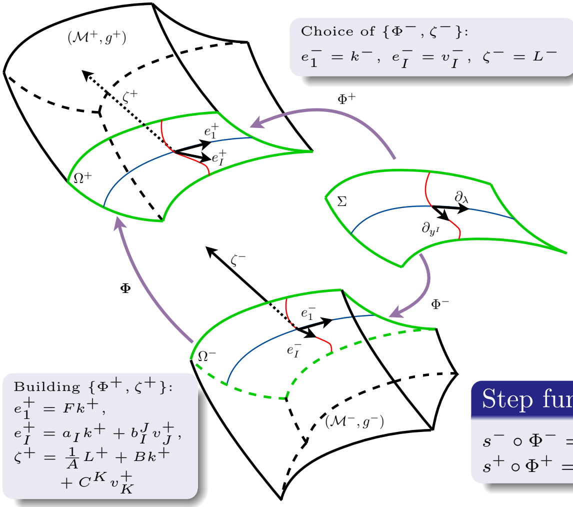
Miguel Manzano, *miguelmanzano06@usal.es*
Marc Mars, *marc@usal.es*

Universidad de Salamanca
Instituto Universitario de Física Fundamental y Matemáticas

# General matching across Killing horizons of order zero

## Preliminaries




Choice of $\{\Phi^-, \zeta^-\}$:
$e_1^- = k^-, \; e_I^- = v_I^-, \; \zeta^- = L^-$

Building $\{\Phi^+, \zeta^+\}$:
$$e_1^+ = F k^+,$$
$$e_I^+ = a_I k^+ + b_I^J v_J^+,$$
$$\zeta^+ = \frac{1}{A} L^+ + B k^+ + C^K v_K^+$$

- $\{\Sigma, \gamma, \boldsymbol{\ell}, \ell^{(2)}, Y^\pm\}$, $n$-dimensional hypersurface data (Mars, 2013) embedded on two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with embeddings $\Phi^\pm$ $(\Omega^\pm := \Phi^\pm(\Sigma))$ and riggings $\zeta^\pm$
- $\{L^\pm, k^\pm, v_A^\pm\}$ basis of $\Gamma(T\mathcal{M}^\pm)|_{\Omega^\pm}$ ($k^\pm$ future null generator, $L^\pm$ future and transverse, $v_A^\pm$ spacelike)
- Assumption: $\exists$ foliation defining functions $s^\pm$ such that $k^\pm(s^\pm) = 1$ on $\Omega^\pm$
- $h^\pm$ induced metric on the leaves
- $\{\zeta^\pm, e_a^\pm\}$ to be identified in the matching process
- The whole matching depends on the set of functions $\{H(\lambda, y^B), h^A(y^B)\}$ $(\det(\partial_{y^B} h^A) \neq 0)$

## Step function $H$

$$s^- \circ \Phi^- = \lambda + \text{const.}$$
$$s^+ \circ \Phi^+ = H + \text{const.}$$

## Shell junction conditions

$h^-_{IJ}|_p = b_I^L b_J^K h^+_{LK}|_{\Phi(p)}$, $A > 0$
$A$, $a_I$, $b_I^J$, $F$, $B$, $C^K$ in terms of $\{H, h^A\}$

# Matching across Killing horizons of order zero

After having analyzed the necessary and sufficient conditions that allow for the matching of two spacetimes with null embedded hypersurfaces as boundaries (Manzano-Mars, 2021), we address the problem of matching across Killing horizons of zero order when the symmetry generators are to be identified (Manzano-Mars, arXiv preprint 2205.08831).

**Definition.** *(Killing horizon of order zero, KH$_0$) Let $(\mathcal{M}, g)$ be a spacetime, $\Omega \subset \mathcal{M}$ be a smooth connected null hypersurface without boundary and $\xi \in \Gamma(T\Omega)$ a null vector (symmetry generator). Define $\mathcal{S} := \{p \in \Omega \mid \xi|_p = 0\}$. Then $\mathscr{H}_0 := \Omega \setminus \mathcal{S}$ is a KH$_0$ if:*

(a) $\mathcal{S}$ is the union of smooth connected closed submanifolds of codimension two

(b) $\mathscr{H}_0$ is totally geodesic

The **surface gravity** $\kappa_\xi$ of a KH$_0$ is defined on $\mathscr{H}_0$ by $\operatorname{grad}\left(\langle \xi, \xi \rangle_g\right) \overset{\mathscr{H}_0}{=} -2\kappa_\xi \xi$

- Assumption: $\kappa_\xi \geq 0$ constant on $\mathscr{H}_0$ (extended to $\overline{\mathscr{H}_0}$ as the same constant)
- By definition of KH$_0$, $\xi = Fk$ ($k$ affine from now on)
  - $\nabla_\xi \xi \overset{\mathscr{H}_0}{=} \kappa_\xi \xi$ entails $\xi = (f + \kappa_\xi s)\, k$, with $k(f) = 0$ and $s$ foliation defining function
- Killing horizons $\mathscr{H}$ such that $\overline{\mathscr{H}}$ is a smooth connected hypersurface are KH$_0$

## Fixed point set $\mathcal{S}$

- If $\kappa_\xi|_{\overline{\mathscr{H}}} \neq 0$ and $\mathcal{S} \neq \emptyset$, then $\mathcal{S} := \{p \in \overline{\mathscr{H}_0} \mid s(p) = -\frac{f(p)}{\kappa_\xi}\}$ spacelike
- If $\kappa_\xi|_{\overline{\mathscr{H}}} = 0$, $\mathcal{S}$ empty or the union of codim-2 null subsets of $\overline{\mathscr{H}}$ (zeros of $f$)

# Matching across Killing horizons of order zero

## Matching across $\mathrm{KH}_0$: symmetry generators identified

- Aim: matching of two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with boundaries $\overline{\mathscr{H}_0}^\pm$ when $\xi^\pm$ are identified up to a constant
- Map $\mathbf{\Phi} : \overline{\mathscr{H}_0}^- \longrightarrow \overline{\mathscr{H}_0}^+$ satisfies $\mathbf{\Phi}_\star (\xi^-) |_{\overline{\mathscr{H}_0}^+} = a\xi^+|_{\overline{\mathscr{H}_0}^+}$, $a \in \mathbb{R} - \{0\}$ ($\xi^-$, $a\xi^+$ must be both future or past)
- The submanifolds $\mathcal{S}^\pm$ must be mapped to each other via $\mathbf{\Phi}$
- $\overline{\mathscr{H}_0}^\pm$ are totally geodesic, but the step function $H$ is restricted
- Away from the $\mathcal{S}^\pm$, step function is determined up to an integration function: $f^- + \kappa_\xi^- \lambda = \dfrac{a(f^+ + \kappa_\xi^+ H)}{\partial_\lambda H}$
- In (Manzano-Mars, arXiv preprint 2205.08831), we study the cases: $\kappa_\xi^\pm = 0$, $\kappa_\xi^\pm \neq 0$, $\kappa_\xi^+ = 0$, $\kappa_\xi^- \neq 0$

## Matching across $\mathrm{KH}_0$: $\xi^\pm$ degenerate (i.e. $\kappa_\xi^\pm = 0$)

- $\mathcal{S}$ empty or the union of $m \in \mathbb{N}^*$ smooth connected codim-2 null submanifolds of $\overline{\mathscr{H}_0}$ $\Longrightarrow$ $m^+ = m^-$
- Step function: $H(\lambda, y^A) = \frac{af^+(y^A)}{f^-(y^A)}\lambda + \mathcal{H}(y^A)$, $\mathcal{H}(y^A)$ integration function encoding the matching freedom
- Most general shell has vanishing pressure

## Freedom

- Velocity along null generators of $\overline{\mathscr{H}_0}^\pm$ is totally determined (outside of $\mathcal{S}$) by the identification of $\{\xi^-, a\xi^+\}$
- One still can select any pair of sections, one on each side, and identify them
- The arbitrary function $\mathcal{H}(y^A)$ accounts for this freedom

$$\mathscr{H}_{\rm p}^{\pm} := \left\{f^{\pm} + \kappa_{\xi}^{\pm} s^{\pm} < 0\right\}, \quad \mathscr{H}_{\rm f}^{\pm} := \left\{f^{\pm} + \kappa_{\xi}^{\pm} s^{\pm} > 0\right\}, \quad \mathcal{S}^{\pm} := \{f^{\pm} + \kappa_{\xi}^{\pm} s^{\pm} = 0\} \ (\text{If } \mathcal{S}^{\pm} \neq \emptyset)$$

## Matching across $\text{KH}_0$: $\xi^{\pm}$ non-degenerate (i.e. $\kappa_{\xi}^{\pm} \neq 0$, $\overline{\mathscr{H}_0}^{\pm} \equiv \mathscr{H}_{\rm p}^{\pm} \cup \mathcal{S}^{\pm} \cup \mathscr{H}_{\rm f}^{\pm}$)

- No assumption on the geodesic completeness of $\overline{\mathscr{H}_0}^{\pm}$, non-zero constant $\widehat{\kappa} := a\kappa_{\xi}^{+}(\kappa_{\xi}^{-})^{-1}$

## $\xi^{\pm}$ non-degenerate: case $\mathcal{S}^{\pm} \neq \emptyset$

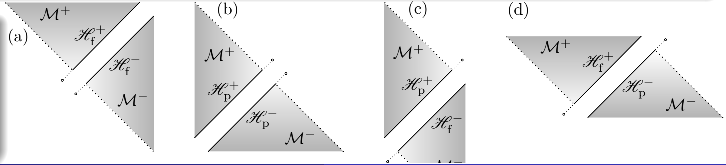
- Step function: $H(\lambda, y^A) = \alpha(y^A)\left(\lambda + \frac{f^{-}(y^A)}{\kappa_{\xi}^{-}}\right) - \frac{f^{+}(y^A)}{\kappa_{\xi}^{+}}$, $\quad \alpha(y^A) > 0$ integration function
- Matching requires $\widehat{\kappa} := a\kappa_{\xi}^{+}(\kappa_{\xi}^{-})^{-1} = 1$, hence surface gravities of $\{\xi^{-}, a\xi^{+}\}$ must coincide
- Resulting shells have vanishing pressure (step function linear in $\lambda$)

## $\xi^{\pm}$ non-degenerate: case $\mathcal{S}^{\pm} = \emptyset \implies \widehat{\kappa}$ not necessarily equal to 1, $\epsilon := \text{sign}\,(\hat{\kappa})\,\text{sign}(f^{-} + \kappa_{\xi}^{-}\lambda)$

- Step function: $H\left(\lambda, y^A\right) = \dfrac{1}{\kappa_{\xi}^{+}}\left(\epsilon\alpha\left(y^A\right)\left|f^{-}\left(y^A\right) + \kappa_{\xi}^{-}\lambda\right|^{\hat{\kappa}} - f^{+}\left(y^A\right)\right)$, $\quad \alpha\left(y^A\right) > 0$, $\quad$ Matchings (a)-(d)

## Freedom associated to $\alpha(y^A)$

- $\mathcal{S}^{\pm} = \emptyset$: freedom of selecting a section on each side and identify them
- $\mathcal{S}^{\pm} \neq \emptyset$: freedom of selecting the initial velocities at $\mathcal{S}^{\pm}$



# Matching across Killing horizons containing bifurcation surfaces

$\mathbf{Ric}^{\pm}$ ambient Ricci, $\mathring{\mathbf{Ric}}^{\pm}$ Ricci on the leaves $\{s^{\pm} = \text{const.}\} \subset \mathscr{H}^{\pm}$, $f_\lambda : S_\lambda \hookrightarrow \Sigma$ embedding

$\widetilde{\boldsymbol{w}}^{\pm} := 2 f_\lambda^* ((\Phi^{\pm})^* (\boldsymbol{\sigma}_{\boldsymbol{L}}^{\pm}))$ (torsion one-form of the sections), $\widetilde{\boldsymbol{R}}^{\pm} := f_\lambda^* ((\Phi^{\pm})^* (\mathbf{Ric}^{\pm}))$, $\boldsymbol{R}^{\parallel} := f_\lambda^* ((\Phi^{\pm})^* (\mathring{\mathbf{Ric}}^{\pm}))$

## Theorem ($\nabla^{\parallel}$ Levi-Civita connection on $\{\lambda = \text{const.}\}$) ($\kappa_\xi^+ = \kappa_\xi^-$) (All dependence in $\lambda$ is explicit)

Let $\mathscr{H}^{\pm}$ be non-degenerate Killing horizons containing bifurcation surfaces $\mathcal{S}^{\pm}$ and $\alpha = \partial_\lambda H$. Then, $Y^{\pm}$ and $\tau$ can be expressed in terms of the tensors

$$\widetilde{\varsigma}_I^{\,-} := \widetilde{w}_I^{\,-}, \qquad \widetilde{\varsigma}_I^{\,+} := \widetilde{w}_I^{\,+} - 2\frac{\nabla_I^{\parallel}\alpha}{\alpha}, \qquad \widetilde{\Xi}_{IJ}^{\pm} := \frac{1}{2}\left( \widetilde{R}_{IJ}^{\pm} - R_{IJ}^{\parallel} - \frac{1}{2}\left( \nabla_I^{\parallel}\widetilde{\varsigma}_J^{\,\pm} + \nabla_J^{\parallel}\widetilde{\varsigma}_I^{\,\pm} \right) + \frac{1}{2}\widetilde{\varsigma}_I^{\,\pm}\widetilde{\varsigma}_J^{\,\pm} \right),$$

as $\quad Y_{11}^- = 0, \quad Y_{1J}^- = -\dfrac{\widetilde{\varsigma}_J^{\,-}}{2}, \quad Y_{IJ}^- = \widetilde{\Xi}_{IJ}^-\lambda; \qquad Y_{11}^+ = 0, \quad Y_{1J}^+ = -\dfrac{\widetilde{\varsigma}_J^{\,+}}{2}, \quad Y_{IJ}^+ = \widetilde{\Xi}_{IJ}^+\lambda \quad$ and

Shell's energy-momentum tensor $\tau$: $\quad \tau^{11} = -\gamma^{IJ}[\widetilde{\Xi}_{IJ}]\lambda, \qquad \tau^{1I} = -\dfrac{1}{2}\gamma^{IJ}[\widetilde{\varsigma}_J], \qquad \tau^{IJ} = 0$

## Conclusions

- Matter-content given by $\alpha$, geometry of $\mathcal{S}^{\pm}$ and $\widetilde{\boldsymbol{R}}^{\pm}$
- Components $Y_{ab}^{\pm}$, $\tau^{ab}$ either constant along generators or linear in $\lambda$
- Energy density $\rho := \tau^{11}$: either identically zero or unavoidably changes sign
- Energy current $j^I := \tau^{1I}$: independent of $\lambda$ ($j^I$ insensitive to the change of sign of $\rho$)
- Surprising behaviour: energy density changes sign, compatible with shell field equations (Barrabés-Israel, 1991)

In (Manzano-Mars, arXiv preprint 2205.08831), results applied to spherical, plane or hyperbolic symmetric spacetimes