New perspectives on the approximate Gravitational Field outside an Axisymmetric isolated rotating matter distribution

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SUMMARY

- **Investigate the non-Kerr contributions to the Gravitational Field of a compact body.**
- **Review the well-known approximation of slow-rotation**
- **Introduce a new approximation: Matter is highly compact, and almost a black hole**
- **This metric can provide initial conditions for Radiation from non-spherical collapse**

Rotating matter with pressure is not well understood in General Relativity

- THE KERR METRIC is the exact solution describing a rotating black hole, and underpins much of Relativistic Astrophysics.
- Attempts to find a corresponding description for sufficiently realistic non-collapsed rotating matter, both inside and outside the body have so far proved futile.

Relevant historical work

- Kerr metric (1963) – Field of a rotating black hole
- Hartle-Thorne (1968) - Framework for describing slow rotation.
- Ernst Equation (1968) – Framework for obtaining all stationary vacuum fields. But none so far conclusively describe the exterior of an isolated matter distribution.
- Teukolsky (1974) – All perturbations of the Kerr metric, both stationary and time-dependent.
- Mars-Senovilla (1998) – Darmois Matching Conditions for Boundary-value problem between matter and vacuum.
- Sarnobat-Hoenselaers (2006), Vera et al (2006), Reina (2015) – Improved framework for describing slow rotation, properly taking into account the Mars-Senovilla conditions.
- Babak-Glampedakis (2006) – 'Quasi-Kerr' metric which describes a rapidly rotating Kerr Black Hole surrounded by oval-shaped cloud of slowly-rotating matter.
- Cabezas et al (2007) - Post Minkowskian approximation
- Frutos-Alfarro (2016, 2019) – Stationary perturbations of the Kerr metric, but using power series expansions for the non-Kerr contributions, relative to the full Kerr part.

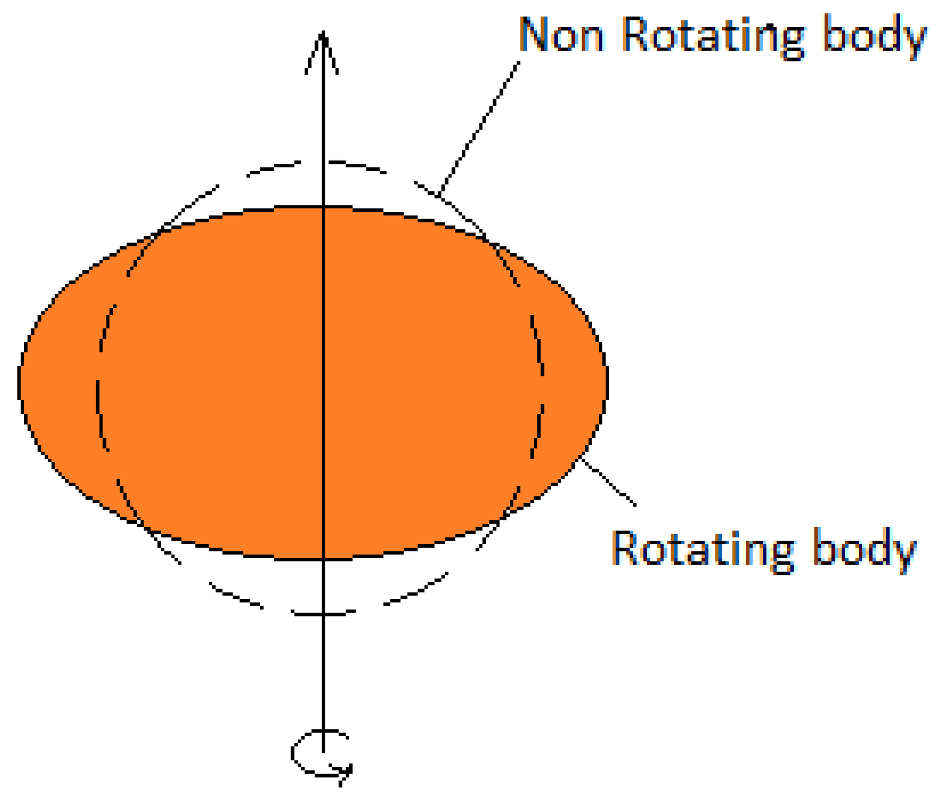
One must resort to Approximation Methods

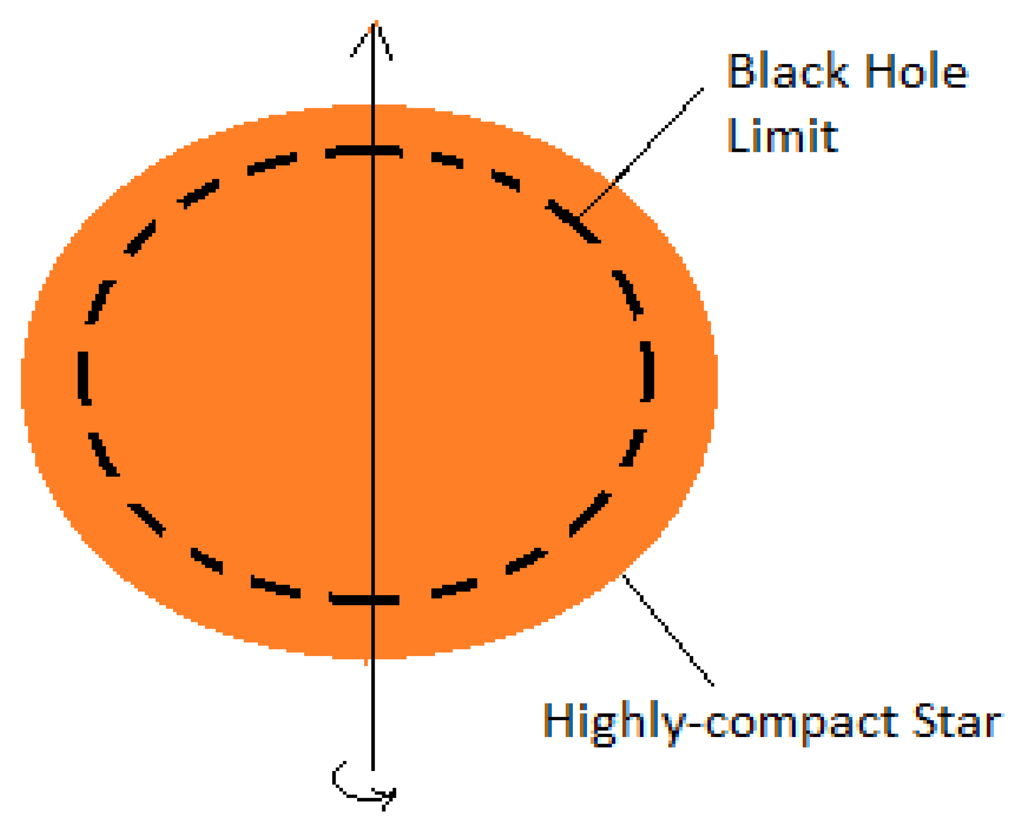
- The full field equations possess strong **non-linearity** and **inter-dependence**. They need to be simplified.
- The post-Minkowskian approximation is one possibility, but this is only valid for small values of compactness.
- The more typical approach is to assume that the rotating star is almost spherical, and only possesses a *small* bulge.
- This would be the case if the star is slowly rotating. In General Relativity, this approximation is popularly described as the **Hartle-Thorne** approximation (1968). Gualtieri (2007) further extended this approximation to intermediate rotation rates.
- A completely different approximation is to assume that the compactness of the star is close to (but not quite) its black hole limit. This approximation was implicitly used by Frutos-Alfaro to specifically describe the exterior of a Neutron Star with any rotation rate (2016, 2019).

Aims of this project

- Review the Hartle-Thorne approximation and the assumptions on which it is based, and discuss variations of it, including the Gualtieri extension.
- Appropriately set up the various frameworks which properly describe perturbations of the Kerr metric. Perform perturbation of both mass and angular momentum parts to first-order in the 'compactness'.
- Compare our results with those of Frutos-Alfaro. Also separately compare with the third-order expansions of Gualtieri, as a limiting case.
- Discuss how our results could be used to describe the start of an axisymmetric collapse under our new approximation.

Comparison of two main approximations:
Left:
Slow-rotation approximation
Right:
Highly-compact approximation





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SLOW ROTATION APPROXIMATION – A REVIEW

- **Hartle-Thorne (1968)**
- **Sarnobat-Hoenselaers (2006)**
- **MacCallum, Mars, and Vera (2006)**
- **Gualtieri et al (2007). Alternative approach to this in the current project.**

To determine the effect of rotation on the shape, second-order expansions are required

- The Dragging of Inertial frames can be obtained by expanding the field equations to first-order in the rotation rate.
- To determine the shape of the boundary, the Hartle-Thorne (H-T) framework expands the field equations to **second-order** in the rotation rate, for both exterior and interior. The former is solved analytically, whereas the latter is dealt with numerically. The solutions are matched at the boundary.
- The second-order perturbed field can be decomposed into a 'smearing' part, and a 'shaping' part. The former would describe what happens if the mass that constitutes the bulge were smeared such that the boundary was 'spherical'. The latter describes the *modulation* of this smearing.
- But H-T is based on a coordinate system that fails to remain invariant during rotation, and therefore the matching fails to properly take into account the symmetries of the gravitational field, and any simplifications that may occur.
- Their results for the vacuum are useful away from the boundary, but *on* the boundary a different approach to matching is required.

Relevant Results

- Ernst Equation $Re(E)\nabla^2 E = (\nabla E)^2$ where $E = F + iA$, and the independent variables x ('radial' coordinate) and y (angular coordinate) are related to Weyl coordinates ρ and z.
- F is the 'Longitudinal potential', A is the 'Dragging potential'
- Expand the Ernst Equation to various orders in the rotation rate.
- Non-rotating: F_0 is what is in the (full) Schwarzschild metric.
- First order: A_1 is that of the Kerr metric to the same order.
- Second order perturbed Ernst Equation, $L_2(E_2) = -(\nabla E_1)^2$, L_2 is an elliptic operator consisting of first and second derivatives.
- Second-order result: $E_2 = Kerr + \sum_{l=0,2} c_l \left[R_l(x) + S_l(x) ln\left(\frac{x-1}{x+1}\right)\right] P_l(y)$ where P_l is a Legendre Polynomial in y, and R_l and S_l are polynomials in x.
- At Third-order, a similar result has been found to hold for E_3, but with Legendre Polynomials l=1 and 3 instead of l=0 and 2.

But slow-rotation is not the only approximation!

The Ernst Potential formalism is naturally adapted to the symmetries of the problem

- A natural coordinate system arises that is a direct statement about the symmetries - **Weyl Coordinates** .
- The equations for the vacuum simplify considerably when compared to their counterparts in the interior. The solving of the Field Equations essentially reduces to that of solving the 'Longitudinal' part of the field, and the Dragging of inertial frames - **Ernst Equation** (1968).
- Similarly, the **Darmois Matching Conditions** for the exterior and interior fields also simplify under this special coordinate system (Mars-Senovilla 1998).
- Sarnobat-Hoenselaers (2006) solved the Ernst Equation for slow rotation to second-order, based on treating a *known* interior (Wahlquist 1968) as a **Boundary-value problem**. Although the *whole* vacuum metric was not Asymptotically Flat, it did nevertheless have a subcase.
- MacCallum et al (2006) also solved the Ernst Equation to second-order for a generic interior, where they developed conditions for asymptotic flatness. Reina (2015) matched this vacuum to the slowly rotating interior in an analogous manner to H-T original scheme.

And to obtain the effect of the object's shape on the dragging of inertial frames, third order expansions are required.

- The Second-order expansions only describe 'mass perturbations' i.e. how much mass ends up in the bulge. For slow rotation rates, this is adequate.
- But at intermediate rotation rates, the shape of the boundary itself is going to contribute to the dragging of inertial frames. Therefore, we must expand the Field Equations to **third-order** in the rotation rate. This describes 'angular momentum perturbations'.
- Gualtieri et al (2007) further extended the H-T framework, for both the interior and exterior. Analytical expressions for the latter were obtained.
- On the other hand, the author has solved the Ernst Equation to third-order in the rotation rate. Relevant matching conditions will be developed in a future work.

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THE HIGHLY-COMPACT APPROXIMATION

- **The exterior of a nearly-collapsed stationary star will be similar to the Kerr metric**
- **Initial attempts by Babak-Glampedakis (2006), Frutos-Alfarro (2016, 2019)**
- **Instead we solve the Teukolsky equation, and then add on a 'completion piece'**
- **This builds on Teukolsky (1974), Sano-Tagoshi (2015), and Van de Meent (2015).**

An alternative to slow-rotation is assuming that the body is almost (but not quite) a black hole

- A Kerr Black Hole is widely believed to be the end state of the collapse of rotating matter. Therefore, one can imagine a situation where the initial (stationary) star has a compactness that is 'almost' that of a black hole. Note that in this approximation, slow rotation is *not* a requirement.
- Actually, it has already been shown that in the non-rotating case the star can only be in equilibrium if its radius is at least 9/8 of the Schwarzschild radius; this is the **Buchdahl limit**. It is not difficult to see that a Buchdahl-like condition must also hold for rotating stars, vindicating our approximation. Also see Neugebauer (2004).
- Unlike the equations describing perturbations of the Schwarzschild metric, the Field Equations describing stationary perturbations of the Kerr metric remain fully coupled, unless additional assumptions are made.
- Babak and Glampedakis (2006) obtained the vacuum field for a full Kerr black hole surrounded by slowly rotating matter.
- Frutos-Alfaro (2016, 2019) assumed a **power-series** expansion for the non-Kerr parts of the metric describing a Neutron Star having any rotation rate.

The Teukolsky Equation describes all possible perturbations of the Kerr metric…

- It was shown by Teukolsky (1974) that the equations describing perturbations of the Kerr metric decouple in an analagous manner that they did for perturbations of the Schwarzschild metric. The Single PDE which characterizes all the perturbations is the **Teukolsky Equation**, where the dependent variable describes the invariant Tidal components of this perturbed field.
- As our proposed vacuum field is 'close' to the Kerr metric, it is reasonable to solve the stationary Teukolsky Equation to describe the non-Kerr part.
- The Teukolsky Equation is widely used in the analysis of gravitational waves from disturbed Kerr Black holes; it has also been occasionally used to describe stationary perturbations of the Kerr metric.
- In particular, Sano-Tagoshi (2015) has already applied this method to describe a Kerr black hole surrounded by a thin ring, while Le Tiec (2020) has investigated tidal deformations of the Kerr event horizon .
- After solving for the rotational parts, we must apply the **Hertz Potential** method (do not confuse with the 'Ernst Potential' method!) to eventually obtain the metric.

Relevant Results

Perturbed Weyl Scalar expansion: $\psi_{(s)} = \sum_{l=2}^{\infty} R_l^{(s)}(r) Y_l^{(s)}(\theta)$

$Y_l^{(s)}(\theta)$ is a spin-weighted Spherical Harmonic, while $R_l^{(s)}$ satisfies the Radial part of the Teukolsky Equation (c.f. Sano-Tagoshi 2015)

$$\left[\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) - \Delta^{s}(l-2)(l-3)\right] R_l^{(s)} = 0$$

where in our case, l=2 (no other l), s= ± 2 and $\Delta = r^2 - 2Mr + a^2$

Our solution for s=2 and s=-2 respectively are:

$$R_{+2}(r) = C_1 + C_2(a^2)tan^{-1}\left(\frac{m-r}{\sqrt{a^2-m^2}}\right) - \frac{C_2(a^2)}{\Delta^2}H(a^2, r^3)$$

$$\text{and } R_{-2}(r) = F_1 + F_2 K(a^2, r^3)$$

$C_1, C_2(a^2)$ and F_1 are constants w.r.t r, while H and K are polynomials up to r^3, while also being functions of a^2.

Perturbed Ernst Equation(s): $L\{F_1\} = \nabla F_0 . \nabla A_0$ and similarly for A_1

…..well, almost. It does not include 'trivial' changes in mass and angular momentum

- The Teukolsky Equation was originally developed for radiative perturbations. Although it does contain a contribution from stationary behavior, this only accounts for the 'shaping' part of the perturbed field and not the 'smearing' part. The latter describes trivial increments in the mass and angular momentum.
- To consider a smearing part, Van de Meent (2015) added a **Completion Piece** to the metric obtained *after* application of the Hertz potential method. We must similarly also search for an appropriate completion piece after applying the Hertz Potential method.
- Then convert the new metric into Weyl Coordinates, and check that it satisfies the perturbed Ernst Equation.
- Finally, compare with Babak-Glampedakis (2006), and the series expansions of Frutos-Alfaro (2016, 2019).

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CONCLUSIONS AND WORK STILL REMAINING

- **Determine Darmois matching conditions for the stationary configuration**
- **The quasi-collapsed metric is the starting point for final non-spherical collapse**
- **Solve the time-dependent axisymmetric Teukolsky equation for the vacuum**
- **Paper(s) to be published!**

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Work Still Remaining

- Apply the Hertz Potential method to determine the metric perturbations from the solution of the Teukolsky equation.
- Pursue an appropriate Completion piece and add that on to the above.
- Convert the whole metric into Weyl Coordinates and check that it satisfies the perturbed Ernst Equation.
- Take the limiting case of this result up to third-order in the rotation rate, and compare with our earlier result obtained from directly solving the Ernst Equation in this approximation
- Determine appropriate Darmois matching conditions for our nearly-collapsed approximation. Possibly use 'Horizon penetrating coordinates', Doran (2000), Ruffini (2018)?
- Solve the time-dependent axisymmetric Teukolsky equation, and obtain the resulting metric. Use the above as stationary initial conditions.
- Compare our procedure against the slow-rotation collapse approximation of Price-Cunningham-Moncrief (1978).
- Also compare our results with the (possibly second-order) Post-Minkowskian approach. Construct a 'hybrid' framework that takes into account both approximations in the compactness. This could potentially represent a wide variety of cases!

Conclusions

- The nearly-collapsed approximation for the stationary configurations is complementary to the widely-used slow-rotation approximation, and also to the post-Minkowskian approximation.
- Although it can be used for *any* rotation rate, it does neverthleless require the body to be highly-compact.
- Unlike Frutos-Alfarro (2016, 2019), we do not make the restrictive assumption about representing the non-Kerr part of the metric as a power-series in the radial coordinate.
- And, unlike Babak-Glampedakis (2006), we do not make the restrictive assumption that the non-Kerr part of the metric requires slow-rotation.
- Our result can provide an initial state for axisymmetric rotating collapse at very high compactness, which is a significant emitter of Gravitational Waves.
- When used in combination with the (second-order) post-Minkowskian approximation, this can represent quite a wide range of cases.
- This entire framework can be used as a limiting comparison with Numerical work.
- Paper(s) to be published!

Quote from Einstein:

'We cannot solve our problems with the same thinking that we used when we created them.'