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Equatorial Timelike Circular Orbits around Generic Ultracompact Objects

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Abstract

For a stationary, axisymmetric, asymptotically flat, ultra-compact [*i.e.* containing light-rings (LRs)] object, with a $\mathbb{Z}_2$ north-south symmetry fixing an equatorial plane, we establish that the structure of timelike circular orbits (TCOs) in the vicinity of the equatorial LRs, for either rotation direction, depends exclusively on the *radial* stability of the LRs. Thus, an unstable LR delimits a region of unstable TCOs (no TCOs) radially above (below) it; a stable LR delimits a region of stable TCOs (no TCOs) radially below (above) it.

Circular geodesics on the equatorial plane

We assume a stationary, axisymmetric, asymptotically flat, 1+3 dimensional spacetime, $(\mathcal{M}, g)$, describing an ultracompact object that may, or may not, have an event horizon.

Using the symmetries together with some gauge choices (see [1] for more details), one can write a generic metric as,

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \ . \qquad (1)$$

We further impose a north-south $\mathbb{Z}_2$ symmetry, and shall look for circular orbits on the equatorial plane, $\theta = \pi/2$.

Test particle motion in the generic geometry (1) is ruled by the effective Lagrangian (dots denote derivatives with respect to an affine parameter),

$$2\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \xi \ , \qquad (2)$$

where $\xi = -1, 0, +1$ for timelike, null and spacelike geodesics, respectively. Introducing the two integrals of motion associated to the Killing vectors, the energy, E, and the angular momentum, L, we can defined an effective potential $V_\xi(r)$ through the Lagrangian, as,

$$V_\xi(r) \equiv g_{rr}\dot{r}^2 = \xi + \frac{A(r,E,L)}{B(r)} \ , \qquad (3)$$

where $A(r,E,L) \equiv g_{\varphi\varphi}E^2 + 2g_{t\varphi}EL + g_{tt}L^2$, and $B(r) \equiv g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} > 0$. A particle will follow a circular orbit at $r = r^{\rm cir}$ iff the following two conditions are simultaneously obeyed throughout the orbit:

$$V_\xi(r^{\rm cir}) = 0 \ , \qquad (4)$$
$$V'_\xi(r^{\rm cir}) = 0 \ , \qquad (5)$$

where prime denotes radial derivative. Moreover, the *radial* stability of such orbit is determined by the sign of $V''_\xi(r^{\rm cir})$.A positive (negative) value implies a unstable (stable) circular orbit.

TCOs. For timelike particles, $\xi = -1$, condition (4) together with the angular velocity $\Omega = d\varphi/dt = \dot{\varphi}/\dot{t}$, determine the energy and angular momentum for circular orbits in terms of the angular velocity as,

$$E_\pm = -\left[\frac{g_{tt} + g_{t\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}\right]_{r^{\rm cir}} \ , \quad L_\pm = \left[\frac{g_{t\varphi} + g_{\varphi\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}\right]_{r^{\rm cir}} \ , \qquad (6)$$

where we have defined

$$\beta_\pm \equiv \left[-g_{tt} - 2g_{t\varphi}\Omega_\pm - g_{\varphi\varphi}\Omega_\pm^2\right]_{r^{\rm cir}} \ . \qquad (7)$$

Then, the remaining condition (5) yields $\Omega_\pm$ in terms of the derivatives of the metric functions at $r^{\rm cir}$,

$$\Omega_\pm = \left[\frac{-g'_{t\varphi} \pm \sqrt{(g'_{t\varphi})^2 - g'_{tt}g'_{\varphi\varphi}}}{g'_{\varphi\varphi}}\right]_{r^{\rm cir}} \ . \qquad (8)$$

One can study the stability of the TCO by using the above results checking the sign of $V''_{-1}(r^{\rm cir})$.

LRs. For null particles, $\xi = 0$, circular orbits are LRs. Condition (4) is a quadratic equation for the *inverse impact parameter*, $\sigma_\pm \equiv E_\pm/L_\pm$, whose solutions are,

$$\sigma_\pm = \left[\frac{-g_{t\varphi} \pm \sqrt{B(r)}}{g_{\varphi\varphi}}\right]_{\rm LR} \ . \qquad (9)$$

The second condition (5), on the other hand, determines LR's radial coordinate. The stability of the LRs is evaluated by checking the sign of $V''_0(r^{\rm LR})$.

TCOs in the vicinity of LRs

Let's assume the existence of a LR. We wish to determine if TCOs exist in its immediate neighbourhood and whether they are stable or unstable.

First, we connect the description of timelike and null orbits. The connection amounts to observe that LRs are determined by

$$\beta_\pm\big|_{\rm LR} = 0 \ , \ \text{and noting that} \ \ \Omega_\pm\big|_{\rm LR} = \sigma_\pm \ . \qquad (10)$$

Indeed, from (7), the condition $\beta_\pm = 0$ becomes equivalent to (4) with $\epsilon = 0$, and (5), also with $\epsilon = 0$, is solved by virtue of (8).

The function $\beta_\pm$ will guide us in the connection between LRs and TCOs. From the continuity of $\beta_\pm$, one expects that (generically) in the neighbourhood of the LR $\beta_\pm$ may be negative. In that case the energy and angular momentum (6) of a timelike particle become imaginary: such region will *not* contain TCOs.

We will now see that there is always one side in the immediate vicinity of a LR, wherein TCOs are forbidden, whose relative location with respect to the LR depends solely on the stability of the latter. For that we perform a first order Taylor expansion of $\beta_\pm$ around the LR. Using the results regarding $\beta_\pm = 0$, one can show that,

$$\beta_\pm(r) \propto V''_0(r^{\rm LR}_\pm)\delta r + \mathcal{O}(\delta r^2) \ , \quad \delta r \equiv r - r_{\rm LR} \qquad (11)$$

Thus, the sign of $\beta_\pm$, and hence the location of the regions where TCOs are forbidden, is determined by the signs of $V''_0(r^{\rm LR}_\pm)$ (stability of the LR) and δr (upper or lower neighbourhood of the LR).

We can extend this analysis further by study the stability of the region where it is possible to have TCOs. For that, we consider the stability of TCOs by examine $V''_{-1}(r)$. This quantity diverges on LRs, since $E_\pm$ and $L_\pm$ diverge when $\beta_\pm(r^{\rm LR}_\pm) \to 0$, by virtue of (6). Approaching the LR from the side wherein TCOs are allowed ($\beta_\pm > 0$), the stability diverges as,

$$V''_{-1}(r_{\rm LR}) \propto \frac{V''_0(r_{\rm LR})}{\beta_\pm(r_{\rm LR})} \qquad (12)$$

Thus, when approaching the LR from the allowed region: TCOs are unstable (stable) if the LR is unstable (stable).

In the end, we conclude that the *radial* stability of a LR determines the localisation and *radial* stability of TCOs surrounding it. Fig. 1 summarises the main results.

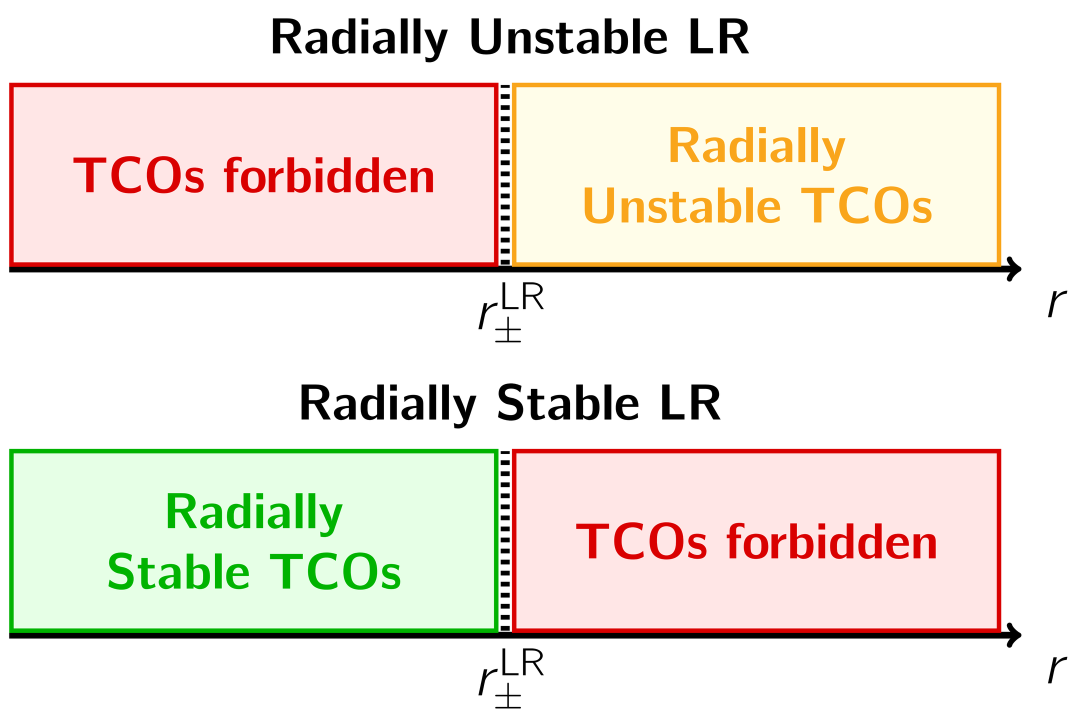

Figure 1: Structure of the equatorial TCOs in the vicinity an unstable (top panel) and stable (bottom panel) LR. Adapted from [1].

References

[1] J. Delgado, C. Herdeiro and E. Radu, *Phys. Rev. D* **105** (2022) 6, 064026