Construction of an energy-momentum tensor for the linearized gravitational field using the Fierz tensor

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Motivated by the problem of finding a satisfactory definition of the energy and momentum of the gravitational field, we propose an energy-momentum tensor for the linearized gravitational field in Minkowski spacetime that has favourable properties (in particular, regarding positivity) and exhibits remarkable similarity to the standard energy-momentum tensor of the electromagnetic field. We obtain this tensor by applying a formulation of linearized gravity based on the Fierz tensor, which can be regarded as a counterpart of the electromagnetic field strength tensor.
For details, references, and further results, see

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Formulation of linearized gravity using the Fierz tensor

Fierz tensor:

$$F_{abc} = \frac{1}{2}(\partial_a h_{bc} - \partial_b h_{ac} + \partial_d h^d{}_a \eta_{bc} - \partial_d h^d{}_b \eta_{ac} - \partial_a h \eta_{bc} + \partial_b h \eta_{ac})$$

(h_{ab} denotes the linearized gravitational field, $h \equiv h^a_a \equiv \eta^{ab} h_{ab}$, η_{ab} denotes the Minkowski metric)
Main properties of F_{abc}:
- invariance under 'scalar' gauge transformations $h_{ab} \to h_{ab} + 2\partial_{ab}\phi$
- antisymmetry in the first two indices: $F_{abc} = -F_{bac}$
- cyclic property: $F_{abc} + F_{bca} + F_{cab} = 0$
- $\partial_c F^c{}_{ab} = -G_{ab}$, $\partial_c F_{ab}{}^c = 0$ (G_{ab} denotes the linearized Einstein tensor)

A useful related tensor:

$$\mathring{F}_{abc} = \frac{1}{2}(\partial_a h_{bc} - \partial_b h_{ac}) = F_{abc} - \frac{1}{2}(F_a \eta_{bc} - F_b \eta_{ac}); \qquad F_a \equiv F_{ab}{}^b = \partial_b h^b{}_a - \partial_a h$$

Lagrangian for the linearized gravitational field:

$$L_1 = \frac{1}{2}(F_{abc} F^{abc} - F_a F^a) = \frac{1}{2} F_{abc} \mathring{F}^{abc}$$

L_1 differs from the usual Fierz–Pauli Lagrangian by a total divergence.

Formulation of linearized gravity using the Fierz tensor

Linearized Einstein equation: $-\partial_c F^{cab} = G^{ab} = \mathcal{T}^{ab}$
(which is understood to be a second order differential equation for h_{ab}).
In topologically simple spacetime domains it is equivalent with the first order equations

$$-\partial_c F^{cab} = \mathcal{T}^{ab}, \qquad \partial_c \tilde{\mathring{F}}^{cab} = 0,$$

where the basic field variable is understood to be F_{abc} (assumed to have the antisymmetry and cyclicity properties), $\mathring{F}_{abc}$ is understood to be defined as $\mathring{F}_{abc} = F_{abc} - \frac{1}{2}(F_a\eta_{bc} - F_b\eta_{ac})$, and $\tilde{}$ denotes duality: $\tilde{F}^{abc} = \frac{1}{2}\epsilon^{abde}F_{de}{}^{c}$.
These equations are similar to Maxwell's equations in their first order form: $\partial_a F^{ab} = \mathcal{J}^b$, $\partial_a \tilde{F}^{ab} = 0$.

Moreover, *in generalized harmonic gauges* ($\partial_a(h^{ab} - \chi\eta^{ab}h) = 0$, $\chi \in \mathbb{R}$) the first order equations can also be written as

$$-\partial_c F^{cab} = \mathcal{T}^{ab}, \qquad \partial_c \tilde{F}^{cab} = 0$$

and F_{abc} satisfies the wave equation

$$\Box F_{abc} = -(\partial_a \mathcal{T}_{bc} - \partial_b \mathcal{T}_{ac}).$$

Energy-momentum tensor

The standard energy-momentum tensor $T^{cd}_{\mathrm{em}} = -F^{ca}F^{d}{}_{a} + \frac{1}{4}\eta^{cd}F_{ab}F^{ab}$ of the electromagnetic field can be obtained by adding a trivially conserved tensor to the canonical energy-momentum tensor. In a similar fashion, by adding the trivially conserved term $-F^{cab}\partial_a h^{d}{}_{b}$ to the canonical energy-momentum tensor that follows from L_1, we obtain

$$T^{cd}_{\mathrm{lg}} = 2\left(F^{cab}\mathring{F}^{d}{}_{ab} - \frac{1}{4}\eta^{cd}F^{eab}\mathring{F}_{eab}\right).$$

In the presence of matter,

$$\partial_c T^{cd}_{\mathrm{lg}} = -2\mathcal{T}^{ab}\mathring{F}^{d}{}_{ab}.$$

(In electrodynamics, $\partial_c T^{cd}_{\mathrm{em}} = -\mathcal{J}^{a}F^{d}{}_{a}$.)

In the $F_a = 0$ gauge, which is a special generalized harmonic gauge, $F_{abc} = \mathring{F}_{abc}$, thus

$$T^{cd}_{\mathrm{lg}} = 2\left(F^{cab}F^{d}{}_{ab} - \frac{1}{4}\eta^{cd}F^{eab}F_{eab}\right).$$

T^{cd}_{lg} shows considerable similarity in the above forms to the usual EM tensor of the electromagnetic field. Similarities can be seen in several other properties as well — see the next page.

Energy-momentum tensor

Main properties of T_{lg}^{cd}:

1. traceless
2. symmetric in the $F_a = 0$ gauge
3. satisfies the dominant energy condition if $F_a = 0$ and $F_{ab0} = 0$, therefore it satisfies this condition in the TT gauge, in particular
4. an expression in F_{abc}, does not depend on higher than first derivatives of h_{ab}
5. invariant under 'scalar' gauge transformations
6. changes by a trivially conserved tensor under general gauge transformations, thus the total energy and momentum it gives are gauge invariant, if suitable fall off conditions at spatial infinity are satisfied
7. for any vector e^a, $T_{\mathrm{lg}}^{cd} e_d$ is a Noether current associated with the variation $\delta h_{ab} = (\mathring{F}_{cab} + \mathring{F}_{cba}) e^c$, which corresponds to an infinitesimal translation in the direction e^a accompanied by a field-dependent infinitesimal gauge transformation
8. gives the same total energy and momentum (up to normalization) as other notable EM tensors (like the linearized Landau–Lifshitz pseudotensor) if h_{ab} and its derivatives fall off sufficiently rapidly at infinity
9. invariant with respect to the duality $F_{abc} \to \tilde{F}_{abc}$ if $F_a = 0$ (this duality exists as a transformation of h_{ab} as well in the $F_a = 0$ gauge and in some subgauges of the $F_a = 0$ gauge, including the TT gauge)