The Validity of the Semiclassical Approximation in 1+1 Electrodynamics: Solutions to the Linear Response Equation

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Abstract

The semiclassical approximation has been used in a wide range of applications in order to obtain interesting physical phenomena. One example explored here considers the Schwinger effect [1], whereby particle production occurs via a classical electric field coupled to a spin ½ field. A natural question which arises is when can this semiclassical approximation be trusted? A criterion for validity of the semiclassical approximation, which originated in semiclassical gravity [2] and later applied to chaotic inflation models [3], is applied here and involves the stability of perturbations to the system characterized by the linear response equation. Solutions to the linear response equation are analyzed here in the context of the validity of the semiclassical approximation. It was found that the semiclassical approximation appears to be valid for electric field strengths much larger than spin ½ particle masses, but invalid for an electric field strength of the same order as the mass.

Introduction

In 1951, Julian Schwinger used the framework of quantum electrodynamics to show, by way of a background field calculation, that pair production occurs due to vacuum decay in the presence of a sufficiently strong classical electric field. To one-loop order, the particle production rate Γ per unit volume×time is expressed as

$$\Gamma = \frac{q^2E^2}{4\pi^3}\sum_{n=1}^{\infty} n^{-2} e^{-\frac{\pi m^2 n}{|q|E}}$$

From this a critical scale for the electric field can thus be defined as $E_{\rm crit} = m^2/q$, above which significant particle production is expected to occur.

There have been extensive efforts to develop a method to analyze when the semiclassical approximation can accurately describe physical systems, summarized in [4]. The method utilized here involves the stability of the electric to linear perturbations. The time evolution of these perturbation is described by the linear response equation. Due to the dependency of the electric field on classical and quantum sources, the linear response equation solution will depend on perturbations in these sources. The perturbation in the source associated with spin ½ pair production involves the vacuum state expectation value of objects constructed from quantized field operators and is thus subject to fluctuations about the average value.

The stability of solutions to the linear response equation against these quantum fluctuations will determine the extent to which the semiclassical approximation is valid. Formally, the criterion for validity of the semiclassical approximation utilized here states: the semiclassical approximation will break down if any linearized, gauge-invariant quantity constructed from solutions to the linear response equation grows rapidly over some period of time. Note, this is a necessary but not sufficient condition for validity.

The objective of this work involves investigating the characteristics of quantum fluctuations and their effect on the solutions to the linear response equation in the context of the validity of the semiclassical approximation.

Semiclassical Model

The action describing a spin ½ field $\psi(t,x)$ coupled to a spatially homogeneous classical background field $A(t)$ is

$$S = \int d^2x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J_C^\mu + i\,\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi \right]$$

Variation of the action yields

$$\frac{\delta S}{\delta A_\mu} = 0 \quad\rightarrow\quad -\Box A^\mu + \partial^\mu\partial_\nu A^\nu = J_C^\mu + J_Q^\mu$$

with spin ½ field current density $J_Q^\mu = q\bar\psi\gamma^\mu\psi$ and two separate classical sources J_C being

$$J_C = -\frac{qE_0}{(1+qt)^2} \quad\rightarrow\quad E_C = E_0\left(\frac{qt}{1+qt}\right)$$

$$J_C = 2qE_0 \operatorname{sech}^2(qt)\tanh(qt) \quad\rightarrow\quad E_C = E_0 \operatorname{sech}^2(qt)$$

The first classical source generates an asymptotically constant electric field for $qt \geq 0$ and the second classical source generates the Sauter pulse with $-\infty \leq qt \leq \infty$. The Dirac equation is arrived at by

$$\frac{\delta S}{\delta\bar\psi} = 0 \quad\rightarrow\quad (i\gamma^\mu D_\mu - m)\psi = 0$$

Expanding the spin ½ field ψ in terms of a complete set of modes

$$\psi(t,x) = \int_{-\infty}^{\infty} dk \left[B_k u_k(t,x) + D_k^\dagger v_k(t,x) \right]$$

with $B_k, B_k^\dagger, D_k, D_k^\dagger$ the creation/annihilation operators for particles and antiparticles obeying the anticommutation relations $\{B_k, B_{k'}\} = \{D_k, D_{k'}\} = \delta(k-k')$. Two independent spinor solutions can be constructed as follows

$$u_k(t,x) = \frac{e^{ikx}}{\sqrt{2\pi}}\begin{pmatrix} h_k^I(t) \\ -h_k^{II}(t)\end{pmatrix} \qquad v_k(t,x) = \frac{e^{-ikx}}{\sqrt{2\pi}}\begin{pmatrix} h_{-k}^{II*}(t) \\ h_{-k}^{I*}(t)\end{pmatrix}$$

Using the Dirac equation, the functions $h_k^{I,II}(t)$ satisfy the mode equations

$$\dot h_k^I - i(k-qA)h_k^I - imh_k^{II} = 0$$

$$\dot h_k^{II} + i(k-qA)h_k^{II} - imh_k^I = 0$$

In the massless limit, these mode equations decouple and have the general solution

$$h_k^{I,II}(t) = \pm\theta(\mp k)e^{i\int_{t_0}^t [k-qA(t')]dt'}$$

The time evolution of the electric field is governed by the semiclassical backreaction equation, obtained by replacing J_Q^μ with its renormalized expectation value $\left\langle J_Q^\mu \right\rangle_{\rm ren}$ and implementing the Lorentz gauge with gauge choice $A^\mu = (0, A(t))$ to yield

$$\ddot A(t) = -\dot E(t) = J_C + \langle J_Q\rangle_{\rm ren}$$

with the vacuum state expectation value

$$\langle J_Q\rangle_{\rm ren} = -\frac{q^2}{\pi}A + \frac{q}{2\pi}\int_{-\infty}^{\infty} dk \left[\left|h_k^I\right|^2 - \left|h_k^{II}\right|^2 + \frac{k}{m} \right]$$

The Linear Response Equation

The criterion for the validity of the semiclassical approximation involves the stability of the system to perturbations. If perturbations grow rapidly over some period of time, the approximation is considered to be invalid. The time evolution of perturbations is governed by the linear response equation. Formally, this can be arrived at by taking a second variation of an effective action $\Gamma[A_\mu]$, obtained by functional integration of the matter degrees of freedom

$$e^{i\Gamma[A_\mu]} = \int D\bar\psi D\psi e^{iS[A_\mu,\bar\psi,\psi]}$$

However, for the system considered here the linear response equation can equivalently by obtained by perturbing the semiclassical backreaction equation about a background solution

$$\delta\ddot A(t) = -\delta\dot E(t) = \delta J_C + \delta\langle J_Q\rangle_{\rm ren}$$

The type of perturbation considered is one which is driven by changes in the classical current. Thus, for the asymptotically constant and Sauter pulse cases respectively one has

$$\delta J_C = -\frac{q}{(1+qt)^2}\delta E_0 \qquad \delta J_C = 2q\operatorname{sech}^2(qt)\tanh(qt)\,\delta E_0$$

The renormalized $\delta\langle J_Q\rangle$ term can be expressed as

$$\delta\langle J_Q\rangle_{\rm ren} = -\frac{q^2}{\pi}\delta A + i\int_{-\infty}^{\infty}dx'\int_{-\infty}^{t}dt'\langle[J_Q(t,x), J_Q(t',x')]\rangle\,\delta A(t')$$

The two-point function can be associated with a generalized susceptibility which is a retarded propagator for the spin ½ field current. Explicitly, the two-point function can be expressed as

$$\langle[J_Q(t,x), J_Q(t',x')]\rangle = \frac{q^2}{2\pi^2}i\int\int dk_1dk_2\, e^{i(k_2-k_1)(x-x')}\operatorname{Im}\{f_{k_1,k_2}(t)g_{k_1,k_2}(t')\}$$

with definitions

$$f_{k_1,k_2}(t) = h_{k_2}^I(t)h_{k_1}^{II}(t) + h_{k_1}^I(t)h_{k_2}^{II}(t)$$

$$g_{k_1,k_2}(t') = h_{k_2}^{I*}(t')h_{k_1}^{II*}(t') + h_{k_1}^{I*}(t')h_{k_2}^{II*}(t')$$

The spatial integral over the two-point function has the form

$$\int_{-\infty}^{\infty}dx'\langle[J_Q(t,x), J_Q(t',x')]\rangle = \frac{4iq^2}{\pi}\int_{-\infty}^{\infty}dk\operatorname{Im}\{h_k^I(t)h_k^{II}(t)h_k^{I*}(t')h_k^{II*}(t')\}$$

Based on the form of the $h_k^{I,II}(t)$ modes in the massless limit, the two-point function returns $\langle[J_Q(t,x), J_Q(t',x')]\rangle = 0$ in this case. This leaves for the linear response equation a simple harmonic form

$$\delta\ddot A(t) + \frac{q^2}{\pi}\delta A = \delta J_C$$

with frequency $|q|/\sqrt{\pi}$, guaranteeing perturbations in this limit remain bounded given the form of δJ_C for the asymptotically constant and Sauter pulse cases considered.

Numerical Results

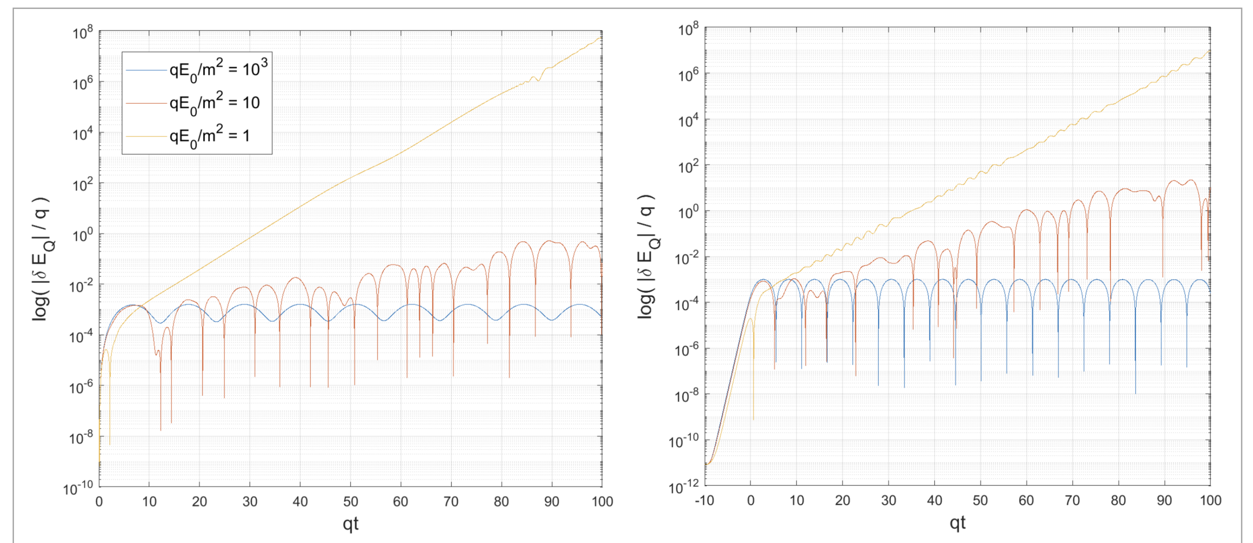

Fig. 1: Numerical solutions for the quantum contribution to the linear response equation δE_Q are plotted for the asymptotically constant case (left) and Sauter pulse case (right). Each plot considers the characteristic quantity qE_0/m^2 for a relatively small, medium, and large electric field strength compared to the spin ½ particle mass, the latter being the critical scale for particle production.

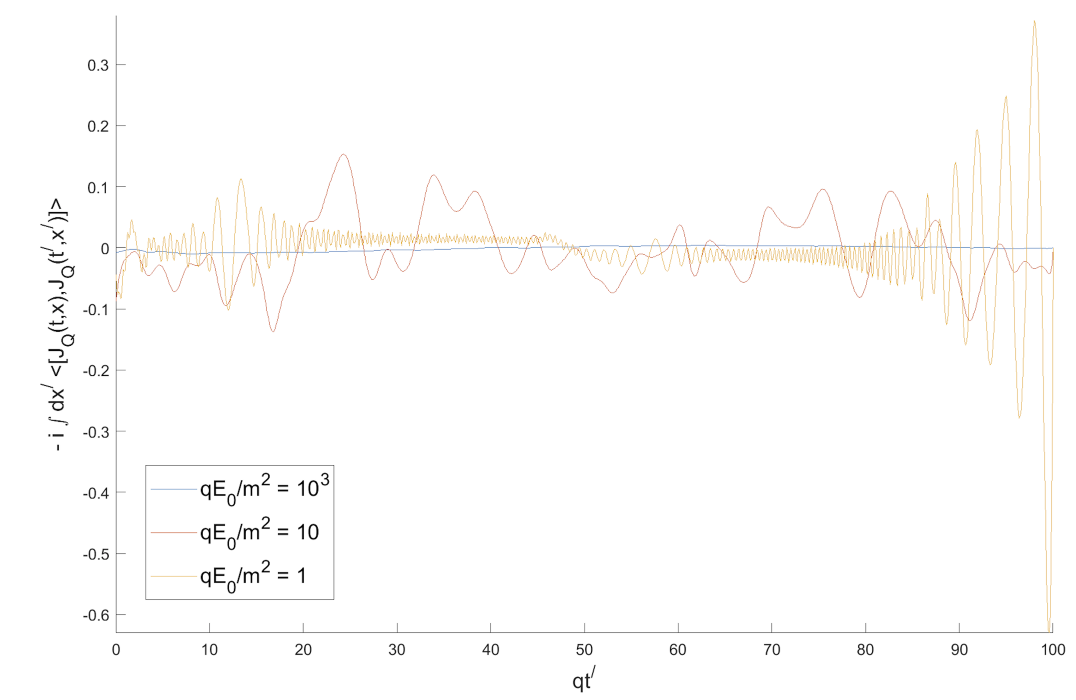

Fig. 2: Preliminary results obtained for the spatial integral over the two-point correlation function for the asymptotically constant case. The value of time qt was chosen as $t = 100$, with the horizontal axis ranging over time t' values. The same cases for the quantity qE_0/m^2 are considered as shown in Fig. 1.

Conclusion

The validity of the semiclassical approximation in 2D electrodynamics has been investigated using a criterion which involves the stability of the system to linear perturbations. If these perturbations grow rapidly over some period of time, the approximation is thought not to be valid. Numerical results indicate the approximation appears to be valid for an electric field strength much larger than spin ½ particle mass ($qE_0/m^2 \gg 1$), but becomes increasingly inaccurate as the critical scale ($qE_0/m^2 = 1$) is approached. This is indicated by unbounded exponential growth in the perturbation solution δE_Q seen in Fig. 1.

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