Mode Decomposing the Second-Order Teukolsky Source

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THE ROYAL SOCIETY

UNIVERSITY OF Southampton

Overview

1. Why decompose the Teukolsky Source?

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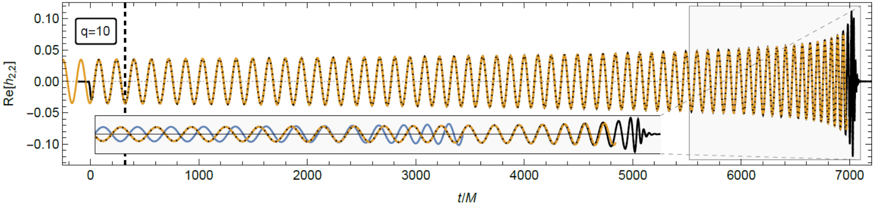
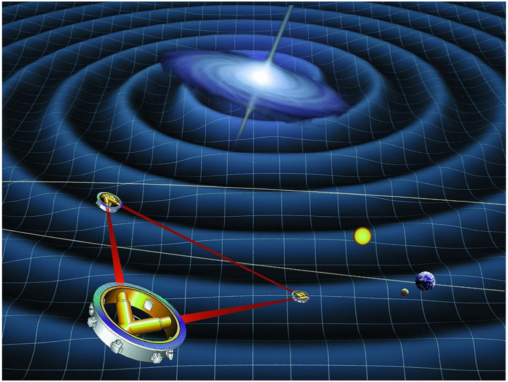
1. Why decompose the Teukolsky Source?
2. **Results:** Second-order Teukolsky source in Schwarzschild
3. Why decomposing the source in Kerr is *TERRIFYING*
4. A method for decomposing in Kerr

Background: Second-order self-force waveforms for quasi-circular inspirals in Schwarzschild



[Wardell et al. arXiv preprint arXiv:2112.12265 (2021).]

Motivation



[Source: NASA, http://lisa.jpl.nasa.gov/gallery/lisa-waves.html.]

Astrophysical supermassive black holes **spin**

Need second-order self-force in **Kerr**

Perturbation theory notation:

$$
\begin{aligned}
g_{ab} =& g^{(0)}_{ab} + \varepsilon^1 h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3), \\
\Rightarrow g_{ab} =& g^{(0)}_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3).
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G_{ab} = \delta G_{ab} + \delta^2 G_{ab} + ...
$$

The separable Teukolsky equation

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The separable Teukolsky equation

$$\mathcal{O}\Big[\psi\Big] = S$$

$$(\mathcal{R} + \Theta)\Big[\psi_{l,m}[r]\ {}_{-2}S_{l,m}[\theta,\phi]e^{i\omega t}\Big] = T_{l,m}[r]\ {}_{-2}S_{l,m}[\theta,\phi]e^{i\omega t}$$

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$$\underline{\mathcal{R}\Big[\psi_{l,m}[r]\Big] = T_{l,m}[r]}$$

Which Teukolsky equation?

First order: $$\mathcal{O}\left[\psi_4^{(1)}\right] = \mathcal{S}\left[T_{ab}^{(1)}\right]$$

$$\Rightarrow \psi = \psi_4^{(1)} := \mathcal{T}[h_{ab}^{(1)}] \quad \& \quad S = \mathcal{S}\left[T_{ab}^{(1)}\right]$$

Which Teukolsky equation?

Second order: $\mathcal{O}\Big[\psi_{4L}^{(2)}\Big] = \mathcal{S}\Big[T_{ab}^{(2)} - \delta^2 G[h_{ab}^{(1)}]\Big]$

$$\Rightarrow \psi = \psi_{4L}^{(2)} := \mathcal{T}[h_{ab}^{(2)}] \text{ \& } S = \mathcal{S}\Big[T_{ab}^{(2)} - \delta^2 G[h_{ab}^{(1)}]\Big]$$

Decomposing means calculating $T_{l,m}[r]$

In Schwarzschild: ${}_{-2}S_{l,m}[\theta, \phi] = {}_{-2}Y_{l,m}[\theta, \phi]$

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$$h_{ab}^{(1)} = h_{ab}^{l,m}[r]\ {}_{s}Y_{l,m}[\theta,\phi]e^{i\omega t}$$

Decomposing means calculating $T_{l,m}[r]$

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$$h_{ab}^{(1)} = h_{ab}^{l,m}[r] \; {}_{s}Y_{l,m}[\theta, \phi] e^{i\omega t}$$

$$-\mathcal{S}\Big[\delta^2 G[h_{ab}^{(1)}]\Big] = T_{l,m}[r] \; {}_{-2}Y_{l,m}[\theta, \phi] e^{i\omega t}$$

Calculating $T_{l,m}[r]$ in Schwarzschild

$$S \sim \partial_\mu \partial_\nu \partial_\alpha \partial_\beta \Big(h_{ab}^{l_1,m_1}[r] \ {}_{s_1}Y_{l_1,m_1} e^{i\omega_1 t} h_{ab}^{l_2,m_2}[r] \ {}_{s_2}Y_{l_2,m_2} e^{i\omega_2 t} \Big)$$

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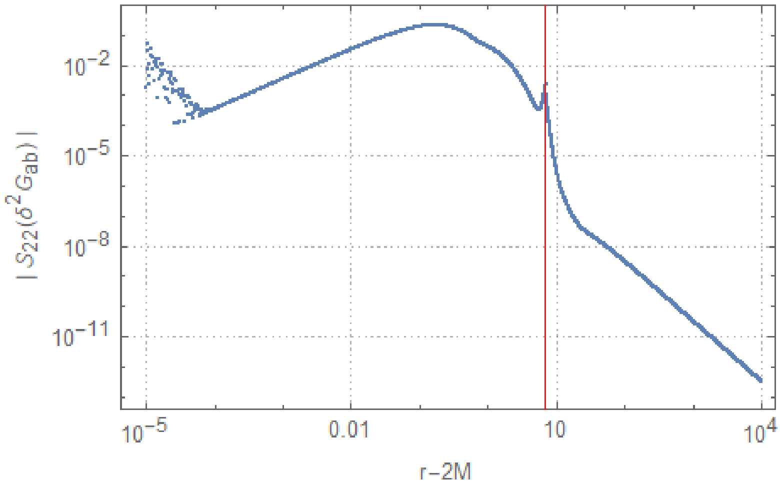
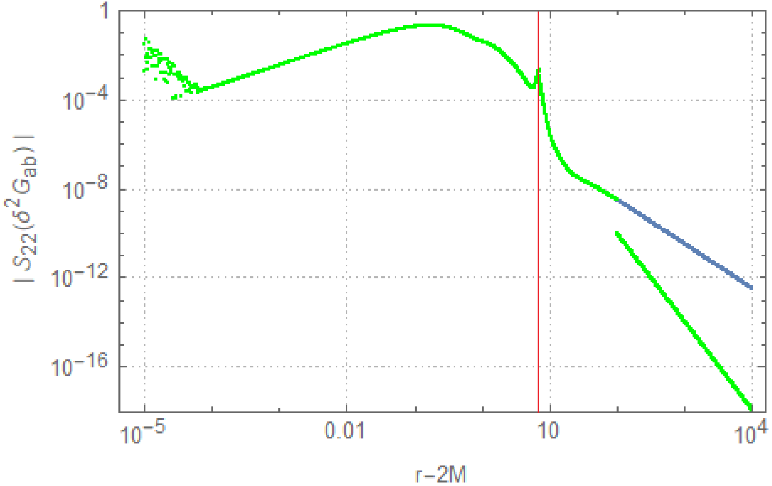
- Combine harmonics algebraically
$$A[r] \ {}_{s_1}Y_{\mathtt{l}_1,\mathtt{m}_1} \ {}_{s_2}Y_{\mathtt{l}_2,\mathtt{m}_2} = A[r] \Big(\oint {}_s\bar{Y}_{l,m} \ {}_{s_1}Y_{l_1,m_1} \ {}_{s_2}Y_{l_2,m_2} d\Omega \Big) \ {}_{-2}Y_{\mathtt{l},\mathtt{m}}$$

- Combine algebraically
$$T_{l,m}[r] = A[r] \Big(\oint \ {}_s\bar{Y}_{l,m} \ {}_{s_1}Y_{l_1,m_1} \ {}_{s_2}Y_{l_2,m_2} d\Omega \Big)$$

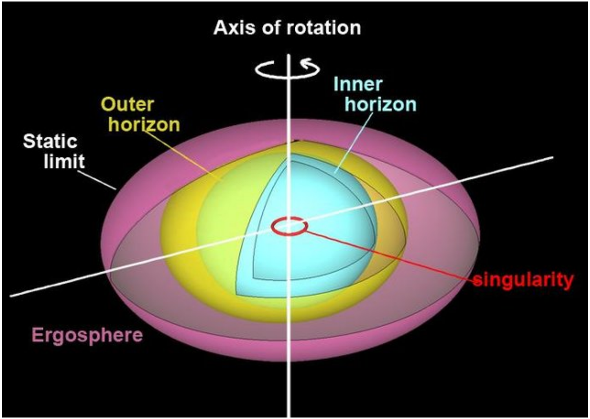
Results: $T_{l=2m=2}[r] = \mathcal{S}\Big[-\delta^2 G[h_{ab}^{(1)}]\Big]_{l=2m=2}$



Near-Bondi–Sachs $T_{l=2m=2}[r] = \mathcal{S}\left[\delta^2 G[h'^{(1)}_{ab}]\right]_{lm}$



The Next Frontier: **Kerr**



K. A. I. L. Wijewardena Gamalath, World Scientific News 114 (2018): 106-125.

Coming up: (*preliminary work*)

- Why decomposing in **Kerr** is more challenging

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- Why decomposing in **Kerr** is more challenging
- How to overcome these problems

What changes from Schwarzschild to Kerr

- ${}_{-2}Y_{l,m}[\theta, \phi] \to {}_{-2}S_{l,m}[\theta, \phi]$ spin-weighted **spheroidal** harmonics

What changes from Schwarzschild to Kerr

- ${}_{-2}Y_{l,m}[\theta, \phi] \to {}_{-2}S_{l,m}[\theta, \phi]$ spin-weighted **spheroidal** harmonics
- $\frac{1}{r} \to \frac{1}{r - ia\cos(\theta)}$

spin-weighted spheroidal harmonics

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- **Do not** have (closed-form [1]) spin-raising/lowering operators ($\hat{\eth}\ {}_sS_{l,m}[\theta,\phi] \neq -\mu[l,s]\ {}_{s+1}S_{l,m}[\theta,\phi]$)

[1] Shah & Whiting, General Relativity and Gravitation 48.6 (2016): 1-19

spin-weighted spheroidal harmonics

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- **Do not** combine algebraically
$${}_{s_1}S_{\mathtt{l}_1,\mathtt{m}_1}\ {}_{s_2}S_{\mathtt{l}_2,\mathtt{m}_2} = \left(\oint\ {}_s\bar{S}_{l,m}\ {}_{s_1}S_{l_1,m_1}\ {}_{s_2}S_{l_2,m_2}d\Omega\right){}_sS_{\mathtt{l},\mathtt{m}}.$$

[1] Shah & Whiting, General Relativity and Gravitation 48.6 (2016): 1-19

Plan:

1. Expand the spheroidal harmonics into spherical harmonics [2-5]

[2] Fallon et al. Physical Review C 81.4 (2010): 041302.
[3] Black Hole Perturbation Toolkit
[4] Hughes. Physical Review D 61.8 (2000): 084004.
[5] Press and Teukolsky. The Astrophysical Journal 185 (1973): 649-674.

Plan:

1. Expand the spheroidal harmonics into spherical harmonics [2-5]
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Plan:

1. Expand the spheroidal harmonics into spherical harmonics [2-5]
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4. Re-expand into spheroidal harmonics [3]

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[3] Black Hole Perturbation Toolkit
[4] Hughes. Physical Review D 61.8 (2000): 084004.
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The *Held* Formalism Summary

All $\partial_\theta \to \hat{\eth}$

[6] Held, Alan. Communications in Mathematical Physics 37.4 (1974): 311-326.

The *Held* Formalism Summary

All $\partial_\theta \to \hat{\eth}$

Collects all $\frac{1}{(r - ia\cos(\theta))^n}$ terms

[6] Held, Alan. Communications in Mathematical Physics 37.4 (1974): 311-326.

Expressing $\frac{1}{(r - ia\cos(\theta))}$: Fourier expansion

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Fourier Expansion (in orders of $\frac{a^2\cos^2(\theta))}{r^2}$)

$y = \frac{1}{(1+\frac{a^2\cos^2(\theta))}{r^2})^{10}}$ Fourier expansion

$(a = 0.9,\ r = r_H,\ n = 10)$

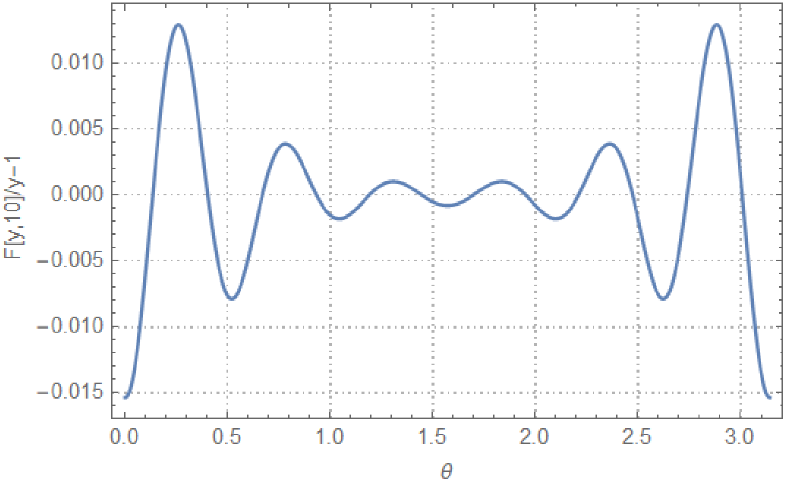

Image credit: Interstellar / R. Hurt / Caltech

Summary

- Second-order Teukolsky source in Schwarzschild obtained
- Decomposition in Kerr will be harder but possible

Thank you for listening

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