Effect of a viscous fluid matter shell on gravitational wave propagation

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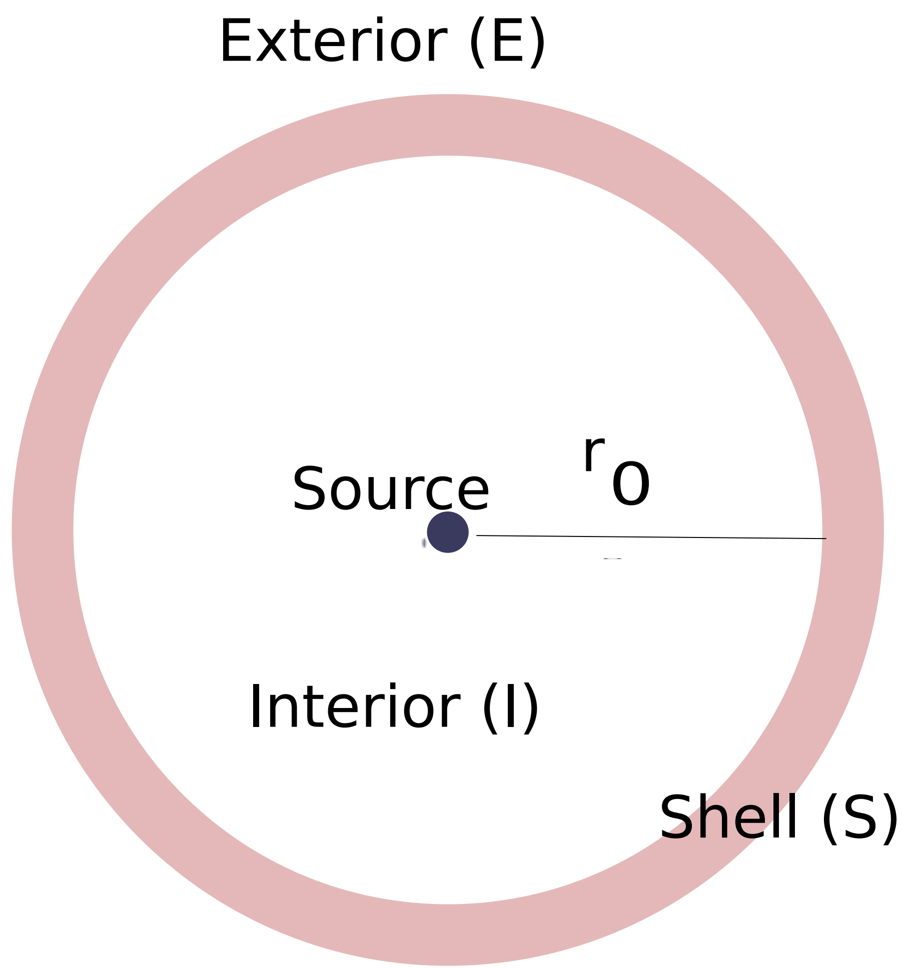

Exterior (E)
Source
r
0
Interior (I)
Shell (S)

Background: Bondi-Sachs Formalism

The Bondi-Sachs metric is

$$ds^2 = -\left(e^{2\beta}(1 + W_c r) - r^2 h_{AB} U^A U^B\right) du^2$$
$$-2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B,$$

where r is an area coordinate so that $det(h_{AB}) = det(q_{AB})$ with q_{AB} a unit sphere metric (e.g. $d\theta^2 + \sin^2\theta d\phi^2$). In Minkowski spacetime $\beta = U^A = 0$, $h_{AB} = q_{AB}$, $W_c = 0$. We introduce a complex dyad q_A (e.g. $q_A = (1, i\sin\theta)$). Then h_{AB}, U^A can be represented by

$$J = h_{AB} q^A q^B / 2 , \quad U = U^A q_A ,$$

with $J \neq 0$ characterizing a deviation from spherical symmetry. In an appropriate gauge, J is directly related to the polarization states of a gravitational wave, $J = h_+ + i h_\times$.

Solution with linearized perturbations

We make the ansatz of a small perturbation about Minkowski spacetime

$$\begin{aligned} \beta =& \Re(\beta^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, \;\; U = \Re(U^{[2,2]}(r)e^{i\nu u})_1 Z_{2,2}\,, \\ W_c =& \Re(W_c^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, \;\; J = \Re(J^{[2,2]}(r)e^{i\nu u})_2 Z_{2,2}\,, \end{aligned}$$

The perturbations oscillate in time with frequency $\nu/(2\pi)$. The quantities ${}_sZ_{\ell,m}$ are "real" spin-weighted spherical harmonic basis functions related to the usual ${}_sY_{\ell,m}$. We consider a source that is continuously emitting GWs at constant frequency dominated by the $\ell = 2$ (quadrupolar) components. Solving the vacuum Einstein equations with no incoming radiation leads to

$$
\begin{aligned}
\beta^{[2,2]} =& b_0\,,\\
W_c^{[2,2]} =& 4i\nu b_0 - 2\nu^4 C_{40} - 2\nu^2 C_{30} + \frac{4i\nu C_{30} - 2b_0 + 4i\nu^3 C_{40}}{r}\\
&+ \frac{12\nu^2 C_{40}}{r^2} - \frac{12i\nu C_{40}}{r^3} - \frac{6C_{40}}{r^4}\,,\\
U^{[2,2]} =& \frac{\sqrt{6}(-2i\nu b_0 + \nu^4 C_{40} + \nu^2 C_{30})}{3} + \frac{2\sqrt{6}b_0}{r} + \frac{2\sqrt{6}C_{30}}{r^2}\\
&- \frac{4i\nu\sqrt{6}C_{40}}{r^4} - \frac{3\sqrt{6}C_{40}}{r^4}\,,\\
J^{[2,2]} =& \frac{2\sqrt{6}(2b_0 + i\nu^3 C_{40} + i\nu C_{30})}{3} + \frac{2\sqrt{6}C_{30}}{r} + \frac{2\sqrt{6}C_{40}}{r^3}\,.
\end{aligned}
$$

Defining the rescaled GW strain by $\mathcal{H}_0 = r(h_+ + ih_\times)$, we find

$$
\mathcal{H}_0 = \Re(H_0 \exp(i\nu u))\,{}_2Z_{2,2} \text{ with } H_0 = -2\sqrt{6}\nu^2 C_{40}\,.
$$

C_{40} is determined by the physics, and b_0, C_{30} are gauge freedoms.

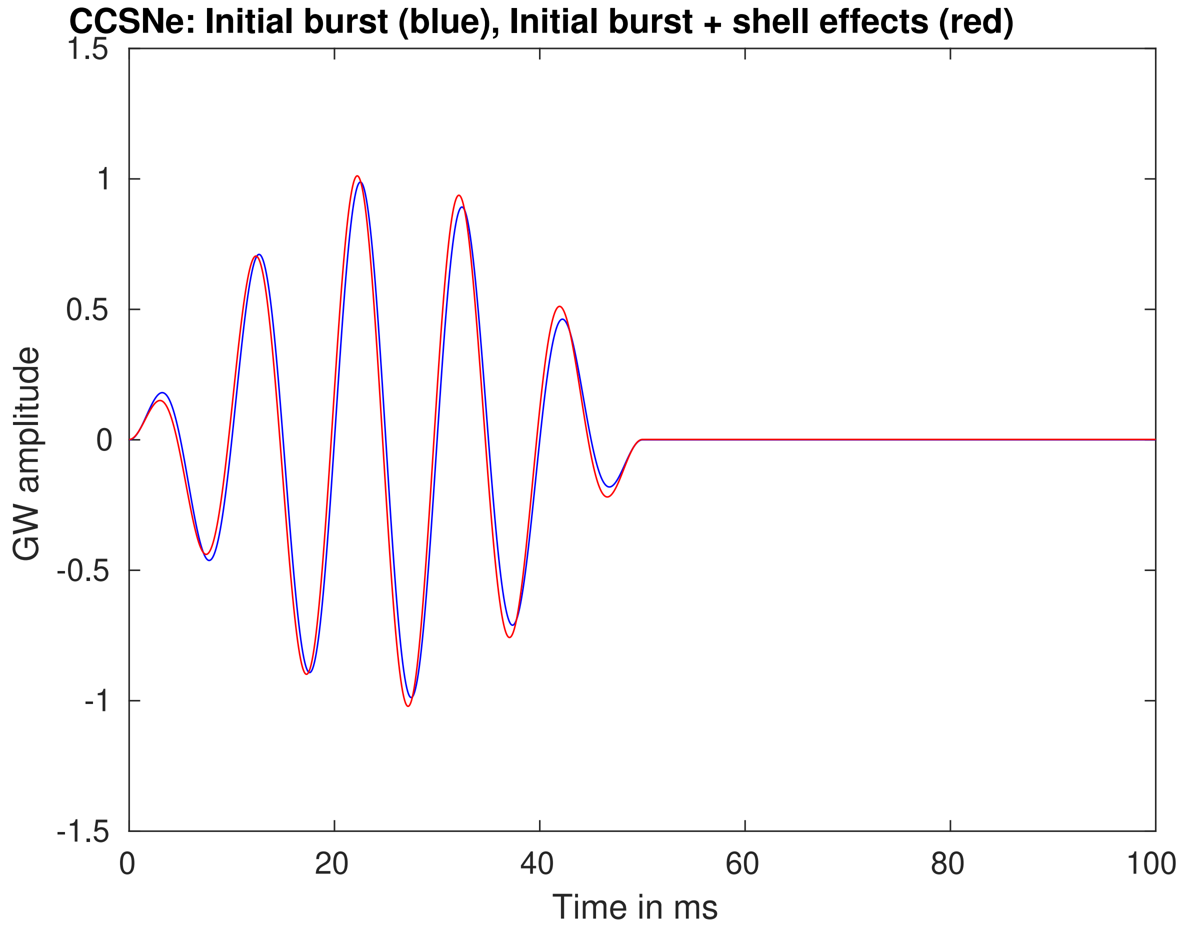
Result for a dust shell

Let M_S be the shell mass, then the GW is

$$H = -2\nu^2\sqrt{6}C_{40}\left(1 + \frac{2M_S}{r_0} - \frac{2iM_S}{r_0^2\nu} + \frac{iM_S e^{-2ir_0\nu}}{2r_0^2\nu} + \mathcal{O}\left(\frac{M_S\delta}{r_0^2}, \frac{M_S}{r_0^3\nu^2}\right)\right) .$$

- Term 1. Effect of matter shell neglected
- Term 2. Effect of gravitational red-shift.
- Term 3. This is out of phase with the leading terms, and thus to $\mathcal{O}(M_S)$, does not affect the magnitude of H.
- Term 4. This affects the magnitude of H, and is due to an incomimg wave modifying the geometry near the source and thus the inspiral rate.

Astrophysical example: Core collapse supernova (CCSNe)



Effect of viscosity: the velocity field

Assuming geodesic motion, the velocity field is

$$V_0^{[2,2]}(r) = \frac{1}{r^3}\times$$
$$\left(3C_{40} - 2ir^3\nu C_{30} - 2ir^3\nu^3 C_{40} + 6i\nu C_{40} r - 2i\nu b_0 r^4 + \nu^2 r^4 C_{30} + \nu^4 r^4 C_{40} - 6C_{40}\nu^2 r^2\right)$$
$$V_1^{[2,2]}(r) = i\frac{9C_{40} + 12i\nu C_{40} r + 2i\nu b_0 r^4 - \nu^2 r^4 C_{30} - \nu^4 r^4 C_{40} - 6C_{40}\nu^2 r^2}{r^4\nu}$$
$$V_{ang}^{[2,2]}(r) = -\frac{i\sqrt{6}}{r^3\nu}\times$$
$$\left(3C_{40} - 2ir^3\nu C_{30} - 2ir^3\nu^3 C_{40} + 6i\nu C_{40} r - 2i\nu b_0 r^4 + \nu^2 r^4 C_{30} + \nu^4 r^4 C_{40} - 6C_{40}\nu^2 r^2\right) .$$

Suppose η is the coefficient of shear viscosity; θ is the fluid expansion, σ_{ab} is the shear tensor, and P_{ab} is the projection tensor:

$$\theta = g^{ab}\nabla_a V_b\,, \quad P_{ab} = g_{ab} + V_a V_b\,,$$
$$\sigma_{ab} = \frac{(P_{ac}\nabla_d V_b + P_{bc}\nabla_d V_a)g^{cd}}{2} - \frac{P_{ab}\theta}{3}\,.$$

We make the usual separation of variables, i.e.,

$$\sigma_{11} = \Re(\sigma_{11}^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, q^A\sigma_{1A} = \Re(\sigma_{1U}^{[2,2]}(r)e^{i\nu u})_1 Z_{2,2}\,,$$
$$q^{AB}\sigma_{AB} = \Re(\sigma_{W}^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, q^A q^B\sigma_{AB} = \Re(\sigma_{J}^{[2,2]}(r)e^{i\nu u})_2 Z_{2,2}\,.$$

We find

$$\theta = \sigma_{00} = \sigma_{01} = \sigma_{0A} = 0\,,$$

$$-\sigma_{11}^{[2,2]} = \sigma_W^{[2,2]} = 12C_{40}\frac{3i - 3r\nu - ir^2\nu^2}{r^5\nu}$$

$$\sigma_{1U}^{[2,2]} = 2C_{40}\frac{6i - 6r\nu - 3ir^2\nu^2 + r^3\nu^3}{r^4\nu}$$

$$\sigma_J^{[2,2]} = C_{40}\frac{-3 - 3ir\nu + 3r^2\nu^2 + 2ir^3\nu^3 - r^4\nu^4}{r^3\nu}\,.$$

The expressions involve only the physical constant C_{40}, and not the gauge freedom constants b_0, C_{30}. Thus σ_{ab} is gauge independent. Also, the fluid expansion $\theta = 0$, so only the shear viscosity has an effect.

Energy loss due to viscosity

We use the formula that the rate of energy loss per unit volume is $-2\eta\sigma_{ab}\sigma^{ab}$. This quantity is evaluated, then integrated over a shell of radius r and thickness δr; the integration is straightforward because of the orthonormality of the angular basis functions. We find that the average rate of energy loss to the shell is

$$\left\langle \dot{E}_\eta \right\rangle = -12\eta C_{40}^2 \nu^6 \delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right),$$

where $\langle f \rangle$ denotes the average of $f(u)$ over a wave period, i.e.

$$\langle f \rangle = \frac{\nu}{2\pi} \int_0^{\frac{2\pi}{\nu}} f\, dt\,,$$

and where we have used $\left\langle \cos^2(\nu u) \right\rangle = \left\langle \sin^2(\nu u) \right\rangle = 1/2$ and $\langle \cos(\nu u) \sin(\nu u) \rangle = 0$.

The rate of energy output as GWs is

$$\left\langle \dot{E}_{GW} \right\rangle = \frac{3C_{40}^2 \nu^6}{4\pi} \,,$$

so that

$$\left\langle \dot{E}_\eta \right\rangle = -16\pi\eta\delta r \left\langle \dot{E}_{GW} \right\rangle \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right) .$$

Energy conservation means that energy absorbed by the viscous fluid is balanced by a reduction in the GW energy. Thus

$$\left\langle \dot{E}_{GW} \right\rangle (r + \delta r) = \left\langle \dot{E}_{GW} \right\rangle (r) \times \left[1 - 16\pi\eta\delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right)\right] .$$

H represents the magnitude of the GWs, and $\left\langle \dot{E}_{GW} \right\rangle \propto H^2$ so

$$H(r + \delta r) = H(r) \left[1 - 8\pi\eta\delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right)\right] .$$

The resulting differential equation is solved to give

$$H(r) = C \exp\left(-8\pi\eta\left(r - \frac{2}{r\nu^2} - \frac{3}{r^3\nu^4} - \frac{9}{r^5\nu^6} - \frac{45}{r^7\nu^8}\right)\right) ,$$

where C is a constant. There are two useful special cases. Let r_i, r_o be the inner and outer radii of the shell. If r_i, r_o are much larger than the wavelength λ of the GWs, then

$$H(r_o) = H(r_i) \exp\left(-8\pi\eta(r_o - r_i)\right) .$$

Equivalent results have been given before*. If r_i is much smaller than the wavelength of the GWs with $r_o \gg r_i$, then

$$H(r_o) = H(r_i) \exp\left(-\frac{360\pi\eta}{r_i^7\nu^8}\right) = H(r_i) \exp\left(-\frac{45\eta\lambda^8}{32 r_i^7 \pi^7}\right) .$$

To our knowledge, viscous damping of GWs with $r_i \ll \lambda$ has not been studied previously, and the result is new.

*E.g., S.W. Hawking, Perturbations of an expanding universe, Astrophys. J. **145**, 544 (1966).

Astrophysical applications

- The formulas above are in geometric units, and are converted to SI units on multiplication η by $G/c^3 = 2.477 \times 10^{-36}$s/kg. Reported values for η may be up to 10^{25}kg/m/s.

- A numerical relativity simulation with GW extraction far from the source would include the above effects. However, in situations such as CCSNe, GW extraction is calculated using a modified quadrupole formula.

- CCSNe includes processes with GW emission at 100Hz, so $\lambda =$3,000km, and $r_i \approx$ 15km. Thus $(\lambda/r_i)^7$ can be large, and GW damping may occur.

- Details will be given in the talk: Session C3, Wednesday July 6, 10:45-11:00, *Damping of gravitational waves originating from core-collapse supernovae*, by Monos Naidoo.

- Primordial GWs may also be damped. Details will be given in the talk: Session B4, Friday July 8, 16:15-16:30, *Cosmological gravitational wave damping as estimated by linearized perturbations on null cone coordinates*, by Petrus van der Walt.

Conclusions

- Changes to both phase and magnitude occur when GWs pass through a shell of matter, and these changes are large enough to be measurable in a GW signal from a close-by source.

- The effect of viscosity on GW propagation was calculated using energy considerations, and the known result for a viscous shell was recovered when $\lambda \ll r_i$.

- However, when $\lambda \gg r_i$, it was found that viscous damping of GWs can be a significant effect.

- In many cases, matter is transparent to GWs, but there are exceptions.

THANK YOU