High-order classical gravitational scattering from quantum scattering amplitudes

Pierre Vanhove

IPhT Saclay

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based on work done in collaboration with
N.E.J. Bjerrum-Bohr, Poul Damgaard, Ludovic Planté, Stavros Mougiakakos

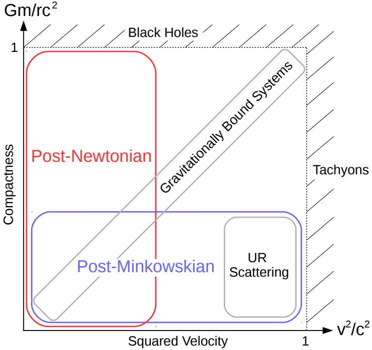
A new area of Precision Gravity

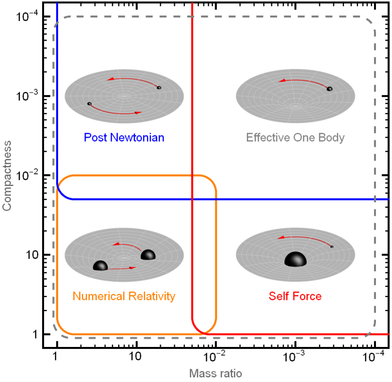
Since the detection of the first gravitational wave signal we entered an area of *precision* gravity.

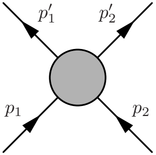
The measurement of the gravitational wave signal is a formidable window on Einstein theory of gravity and possibly beyond. Ultimatly this would tell us how good we understand gravity both in the weakly and strong coupling regimes.

- We need to produce *analytic* formulæ for the theoretical gravitational waveform templates
- We need to understand how much can be understood from exact theoretical computations
  - How much can we understand about strong gravity regime
  - How much can we learn about Gravity beyond Einstein gravity : quantum effects? modified gravity scenarii (extra dimensions, massive gravity, . . .)?

The recent theoretical developments have revealed connections between the scattering amplitudes, classical observables, and gravitational self-force

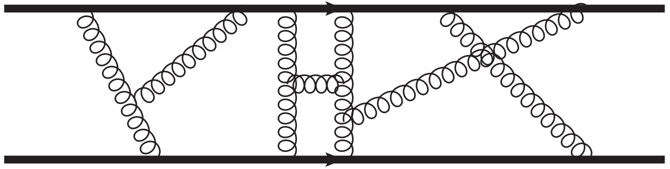






$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}_L(\gamma, m_1, m_2); \qquad \gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

Classical physics from quantum loops



In the limit $\hbar, q^2 \to 0$ with $\underline{q} = \frac{q}{\hbar}$ fixed at each loop order of the quantum amplitude has the Laurent expansion[1] $\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$ and $q^2 = (p_1 - p'_1)^2$

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

- At all loop orders there is a classical contribution of order $1/\hbar$
- The full amplitude has IR and UV divergences, but the classical observables are finite

1 [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove; Kosower, Maybee, O'Connell]

Classical physics from loops : $\hbar$ counting

In QFT the propagator has inverse $\hbar$ that the traditional counting disregards (cf . [Quantum Field Theory, Itzykson Zuber, §6-2-1 page 288])[2]

> To find the connection between L and the power of $\hbar$, we collect all factors $\hbar$. We leave aside the factor $\hbar$ that gives the mass term a correct dimension. In other words, the Klein-Gordon equation should read $[\partial_x^2 + (mc/\hbar)^2]\varphi = 0$, indicating that the mass term is of quantum origin. This phenomenon is disregarded in the sequel. There are thus two origins of such factors. First the

At the $L+1$PM order, the two-body scattering amplitude scales with the masses as

$$\mathcal{M}_L(\gamma, q^2) = \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma) m_1^{L-i} m_2^i$$

This piece will emerge from a L quantum amplitude as follows

$$\mathfrak{M}^L\Big|_{\text{classical}} \propto \frac{m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \hbar^{L-1} G_N^{L+1} \sum_i \left(\frac{m_1 c}{\hbar}\right)^{L-i} \left(\frac{m_2 c}{\hbar}\right)^i \propto \frac{\mathcal{M}_L(\gamma, \underline{q}^2)}{\hbar}$$

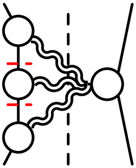
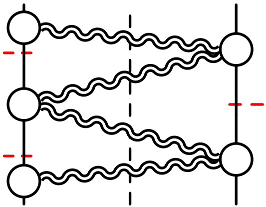
[2] [B. R. Holstein and J. F. Donoghue, [arXiv:hep-th/0405239 [hep-th]].]

Velocity cuts

The mass expansion is then given by

$$\mathcal{M}_L(\gamma, \underline{q}^2) = \frac{G_N^{L+1} m_1^2 m_2^2}{|\underline{q}|^{2-L+\frac{(4-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma, D) m_1^{L-i} m_2^i$$

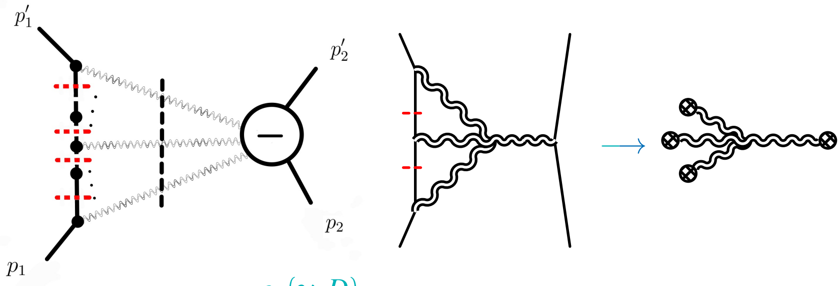
Each nomomials are identified by inserting delta-function *velocity cut* in the amplitude[3]

$m_1^4 m_2^2\, c_{3,1}(\gamma, D) \simeq$  $; m_1^3 m_2^2 c_{3,3}(\gamma, D) \simeq$ 

3 [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Black hole metric

The velocity cut formalism gives the probe regime $m_2 \ll m_1$ contributions to high order in Post-Minkowskian perturbation



$$\mathcal{M}_L^{\text{probe}}(\gamma, \underline{q}^2) = \frac{c_L(\gamma, D)}{(\underline{q}^2)^{1-\frac{(D-3)L}{2}}} m_1 m_2^2 (8\pi G_N m_1)^{L+1},$$

$$\chi^{\text{probe}}(\gamma, D) = \mathcal{N}(L, D) \frac{c_L(\gamma, D)}{(\gamma^2-1)^{L+1}} \left(\underbrace{2\pi^{\frac{3-D}{2}} \frac{\Gamma\left(\frac{D-1}{2}\right)}{(D-2)} \frac{G_N m_1}{b^{D-3}}}_{=\rho(r)} \right)^{L+1}$$

Scattering angle in the black hole metric

The Schwarzschild-Tangherlini metric reads in an isotropic coordinate system

$$ds^2 = \underbrace{\left(\frac{1-\rho(r)}{1+\rho(r)}\right)^2}_{=A(r)} dt^2 + \underbrace{(1+\rho(r))^{\frac{4}{D-3}}}_{=B(r)} \left(dr^2 + r^2 d^2\Omega_{d-2}\right)$$

and the scattering angle

$$\frac{\chi}{2} = -\int_{\hat{r}_m}^{\infty} dr \frac{\partial p_r}{\partial J} - \frac{\pi}{2} = b\int_{\hat{r}_m}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{1-\frac{b^2}{r^2}-\frac{\mathcal{V}_{\text{eff}}(r,E)}{p_\infty^2}}} - \frac{\pi}{2}$$

with the effective potential obtained using Hamilton-Jacobi equations $g^{\mu\nu}\partial_\mu\mathcal{S}\partial_\nu\mathcal{S} = \mu^2$

$$\frac{\mathcal{V}_{\text{eff}}(r)}{p_\infty^2} = 1 - \frac{B(r)}{\gamma^2-1}\left(\frac{\gamma^2}{A(r)}-1\right), \qquad p_r^2 = p_\infty^2 - \mathcal{V}_{\text{eff}}(r,E) - \frac{J^2}{r^2}$$

which matches the one from the amplitude computation in D dimensions

Effective EOB metric

The amplitude set up is very adapted for giving *analytic* input to the EOB formalism.

The EOB effective metric $g^{\text{eff}}_{\mu\nu}$ in isotropic coordinates

$$ds^2_{\text{eff}} = \underbrace{\left(\frac{1-h^{\text{eff}}(r)}{1+h^{\text{eff}}(r)}\right)^2}_{=A^{\text{eff}}(r)} dt^2 - \underbrace{\left(1+h^{\text{eff}}(r)\right)^4}_{=B^{\text{eff}}(r)} \left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

The scattering angle in such an external metric is derived using the principal function and the Hamilton-Jacobi equation

$$\mathcal{S} = \mathcal{E}_{\text{eff}} t + J_{\text{eff}}\varphi + W(r), \qquad g^{\alpha\beta}_{\text{eff}} \partial_\alpha \mathcal{S} \partial_\beta \mathcal{S} = \mu^2$$

The scattering angle

$$\frac{\chi}{2} = J_{\text{eff}} \int_{r_m}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{\frac{B^{\text{eff}}(r)}{A^{\text{eff}}(r)}\mathcal{E}^2_{\text{eff}} - \frac{J^2_{\text{eff}}}{r^2} - B^{\text{eff}}(r)\mu^2}} - \frac{\pi}{2}$$

EOB energy, momentum and angular momentum maps

The effective-one-body formalism is based on the following maps

- The *energy map* $E = (m_1+m_2)\sqrt{1+2\frac{m_1m_2}{(m_1+m_2)^2}\left(\frac{m_1+m_2}{m_1m_2}\mathcal{E}_{\rm eff}-1\right)}$
- The *momentum map* $p_\infty^2 = \frac{(E^2-(m_1+m_2)^2)(E^2-(m_1-m_2)^2)}{4E^2}$, $\qquad \frac{p_{\rm eff}}{\mu} = \frac{p_\infty E}{m_1m_2}$
- An *angular momentum map*

$$b = \frac{J}{p_\infty} = \frac{J_{\rm eff}}{p_{\rm eff}} \Longrightarrow J_{\rm eff} = J\frac{p_{\rm eff}}{p_\infty} = J\,\frac{E}{M}$$

This maps differs from the one used by Damour $J_{\rm eff} = J$

The metric coefficients are then fully determined by the effective potential

$$\frac{\mathcal{V}_{\rm eff}(r,E)}{p_\infty^2} = 1 - \frac{B^{\rm eff}(r)}{\gamma^2-1}\left(\frac{\gamma^2}{A^{\rm eff}(r)}-1\right)$$

With these maps we **never need any non-metric contributions** to the contrary to the "standard" EOB approach of [Buonano, Damour]

Outlook

Many new exciting developpements leading to a better understanding of the gravitational interactions in binary system

- ✓ Improved understanding of the relation between general relativity and the quantum theory of gravity from small velocity to ultra-relativistic regime
- ✓ Clarifying the connection with various approaches for the post-Newtonian, post-Minkowskian, self-force, EOB approach developped other the years
- ✓ New technics that can be applied to efficiently compute high order in the PN or PM expansion
- ✓ Understand the radiation (Radiation-reaction, memory effects,. . .) in a gauge invariant formalism