UNIVERSITY OF CAMBRIDGE
Faculty of Mathematics

A Positive Mass Theorem in Higher Dimensions

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GR 23 Conference, 2022

- *Positivity of Mass in Higher Dimensions (arXiv:2010.05086)*

Outline

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- Discuss rigidity case $m = 0$.

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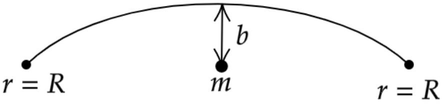

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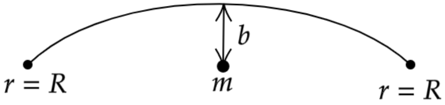

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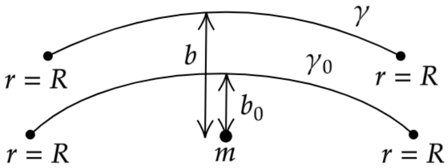

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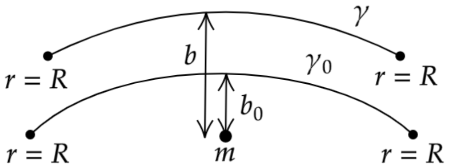

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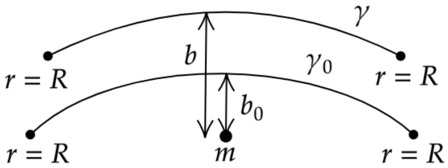

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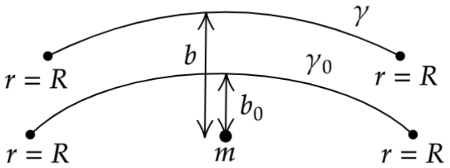

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Focusing theorem $\implies m \geq 0$

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Theorem (Penrose, Sorkin, Woolgar)

The 4 momentum of (M, g) is future causal ($k \cdot P \geq 0$ for all null k). This means the ADM energy is non-negative in every rest frame.

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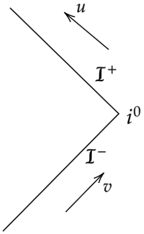

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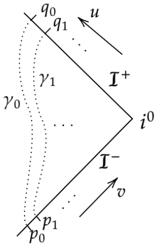

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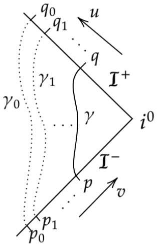

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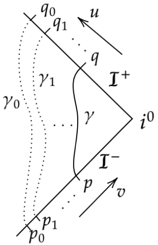

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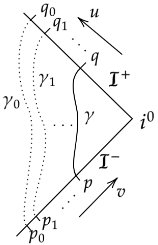

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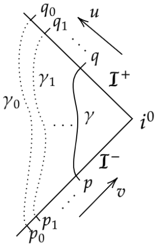

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- γ contains no conjugate points (since it lies on the boundary of $J^+(p)$).

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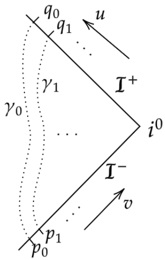

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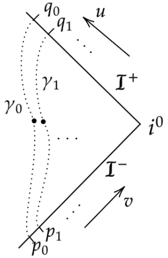

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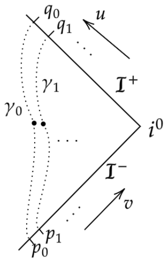

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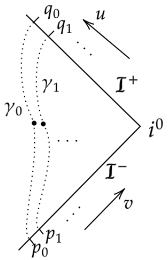

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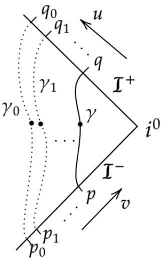

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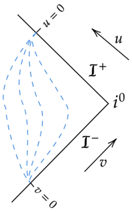

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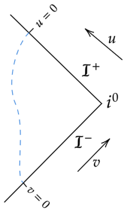

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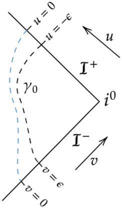

- Pick one which is g-timelike.
- Modify to have negative time of flight.

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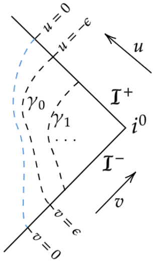

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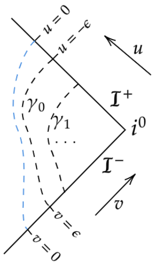

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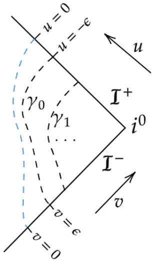

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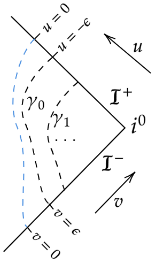

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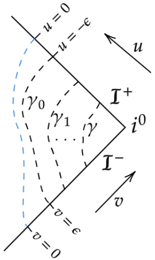

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- In this case, large r behaviour of the metric is controlled by higher order terms over which we have no control.

Questions?