Additively separable family of solutions for the null-surface formulation of general relativity in 2+1 dimensions

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Null-Surface Formulation

- An alternative formulation of general relativity where the central role is played by null surfaces instead of the metric tensor.

- 3+1 version introduced by Frittelli, Kozameh and Newman in 1995[1–3]. They hoped it might provide insight into issues such as gravitational lensing.

- Has been developed in higher dimensions by Gallo[4].

(1) Frittelli, S., Kozameh, C.N., Newman, E.T.: J. Math. Phys. **36**, 4975 (1995)
(2) Frittelli, S., Kozameh, C.N., Newman, E.T.: J. Math. Phys. **36**, 4984 (1995)
(3) Frittelli, S., Kozameh, C.N., Newman, E.T.: J. Math. Phys. **36**, 5005 (1995)
(4) Gallo, E. : Class. Quantum Grav. **29**, 145004 (2012)

Null Surface Formulation

The 2+1 version of the theory was developed by

- Forni, Iriondo, Kozameh and Parisi(5,6)
- Tanimoto(7)
- Silva-Ortigoza(8)

(5) Forni, D.M., Iriondo, M., Kozameh, C.N.: J. Math. Phys. **41**, 5517 (2000)
(6) Forni, D.M., Iriondo, M., Kozameh, C.N., Parisi, M.F.: J. Math. Phys. **43**, 1584 (2002)
(7) Tanimoto, M.: On the Null Surface Formulation –Formulation in Three Dimensions and Gauge Freedom,gr-qc/9703003v1
(8) Silva-Ortigoza, G.: Gen. Relativ. Gravit. **32**, 2243 (2000)

2+1 Null Surface Formulation

There is interest in the 2+1 version because

- the equations of $n + 1$ NSF, $n \geq 3$ have not yet been solved.

  While the equations of the 2+1 version are still difficult to solve, three solutions were found in 2014, 2018 and 2019[(9–11)]

  - 2+1 solutions may suggest methods for finding solutions in higher dimensions.

- there is a connection to Cartan's work on classifying third order ordinary differential equations.[(13–15)]

(9) Harriott, T.A., Williams, J.G.: Gen Relativ. Gravit. **46**, 1666 (2014)
(10) Harriott, T.A., Williams, J.G.: Gen Relativ. Gravit. **50**, 39 (2018)
(11) Harriott, T.A., Williams, J.G.: Gen Relativ. Gravit. **51**, 98 (2019)
(12) Cartan, E.: C.R. Acad. Sci. **206**, 1425 (1938)
(13) Cartan, E.: Rev. Mat. Hispano-Amer. **4**, 1 (1941)
(14) Cartan, E.: Ann. Sc. Ec. Norm. Sup. (3) **60**, 1 (1943)
(15) Chern, S.-S.: The Geometry of the differential equation $y''' = F(x, y, y', y'')$. In Selected Papers, Springer, New York (1978). Original version: Science Reports Nat. Tsing Hua Univ. **4**, 97 (1940)

2+1 Version of NSF: Details

- The families of null surfaces are described by

$$u = Z(x^a; \varphi) = \text{ constant}$$

where x^a are the spacetime coordinates and φ labels the surfaces.

- In order to guarantee that for each choice φ, these surfaces are null surfaces with respect to some spacetime metric $g_{bc}(x^a)$ the gradient of Z must satisfy $g^{bc}\left(x^a\right)Z_{,b}(x^a;\varphi)Z_{,c}\left(x^a;\varphi\right) = 0$ for arbitrary values of the parameter φ and where $Z_{,a} = \partial Z / \partial x^a$.

2+1 Version of NSF: Details

- Conditions that ensure that solutions exist that represent null surfaces are called metricity conditions.

- These metricity conditions are the NSF field equations that must be solved to find the null surfaces.

However, these conditions are more easily found by first introducing new coordinates.

2+1 Version of NSF: Intrinsic Coordinates

- It is convenient to introduce new coordinates called intrinsic coordinates, obtained by repeatedly differentiating u with respect to φ:

$$
\begin{aligned}
u &:= Z(x^a;\varphi),\\
\omega &:= \partial Z(x^a;\ \varphi),\\
\rho &:= \partial^2 Z(x^a;\varphi)
\end{aligned}
$$

where ∂ denotes differentiation with respect to φ with the x^a held fixed.

- Using the intrinsic coordinates the metricity conditions are most simply expressed in terms of a function

$$\Lambda\left(u,\omega,\rho,\varphi\right)=\partial^3 Z\left(x^a;\varphi\right)=\partial^3 Z\left(x^a\left(u,\omega,\rho,\varphi\right);\varphi\right)$$

and a second auxiliary function $\Omega\left(u,\omega,\rho,\varphi\right)$.

2+1 Version of NSF: Summary

In these coordinates the metricity conditions are three coupled equations that must be solved to find Λ and Ω[5]:

(I) $$2\left[\eth(\partial_\rho\Lambda) - \partial_\omega\Lambda - \frac{2}{9}(\partial_\rho\Lambda)^2\right](\partial_\rho\Lambda) - \eth^2(\partial_\rho\Lambda) + 3\eth(\partial_\omega\Lambda) - 6\partial_u\Lambda = 0$$

(Main Metricity Condition - MMC)

where $\eth = \partial_\varphi + \omega\partial_u + \rho\partial_\omega + \Lambda\partial_\rho$

(5) Forni, D.M., Iriondo, M., Kozameh, C.N.: J. Math. Phys. **41**, 5517 (2000)

2+1 Version of NSF: Summary

In these coordinates the metricity conditions are three coupled equations that must be solved to find Λ and $\Omega^{(5)}$:

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(Main Metricity Condition - MMC)

where $\eth=\partial_\varphi+\omega\partial_u+\rho\partial_\omega+\Lambda\partial_\rho$

(II) $$3\eth\Omega=\Omega\partial_\rho\Lambda$$

(Secondary Metricity Condition)

Together these two expressions guarantee that the surfaces determined by Λ will be null surfaces with respect to some spacetime metric $g_{bc}\left(x^a\right)$.

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Together these two expressions guarantee that the surfaces determined by Λ will be null surfaces with respect to some spacetime metric $g_{bc}\left(x^a\right)$.

(III) $$\partial_\rho^2\Omega=\kappa T_{\rho\rho}\Omega$$

If this third equation is satisfied along with (I) and (II) then so are the 2+1 Einstein equations[5].

(5) Forni, D.M., Iriondo, M., Kozameh, C.N.: J. Math. Phys. **41**, 5517 (2000)

Finding Solutions

Consider the main metricity condition

$$2\left[\partial\left(\partial_{\rho}\Lambda\right)-\partial_{\omega}\Lambda-\frac{2}{9}\left(\partial_{\rho}\Lambda\right)^{2}\right]\left(\partial_{\rho}\Lambda\right)-\partial^{2}\left(\partial_{\rho}\Lambda\right)+3\partial\left(\partial_{\omega}\Lambda\right)-6\partial_{u}\Lambda=0.$$

The form of the operator $\partial=\partial_{\varphi}+\omega\partial_{u}+\rho\partial_{\omega}+\Lambda\partial_{\rho}$

suggests looking for additively separable solutions for $\Lambda\left(u,\omega,\rho\right)$.

Specifically, choosing

$$\Lambda\left(u,\omega,\rho\right)=-a\omega+h\left(\rho+au\right)=-a\omega+h\left(x\right)$$

for constant a and writing $x=\rho+au$, the main metricity condition is converted into an ordinary differential equation in *x*.

Solution of Main Metricity Condition

With this choice the main metricity condition

becomes $$h^2 \frac{d^3h}{dx^3} - h\frac{dh}{dx}\frac{d^2h}{dx^2} + \frac{4}{9}\left(\frac{dh}{dx}\right)^3 + 4a\frac{dh}{dx} = 0$$

Writing $y = h^{2/3}$ allows the metricity condition to be integrated three times so that y can be expressed as an implicit function of x:

$$x = \pm\int \frac{ydy}{\sqrt{y\left(2ky^2 - Ay - 4a\right)}}$$

where the k and A are constants arising from those integrations.

It represents an exact solution of the main metricity condition.

That is, it gives an expression for $\Lambda\left(u,\omega,\rho\right)$ which was chosen to have the form

$$\Lambda\left(u,\omega,\rho\right) = -a\omega + h\left(\rho + au\right) = -a\omega + y^{3/2}\left(x\right)$$

for some constant a and where $x = \rho + au,\ y = h^{2/3}$.

Solution of Second Metricity Condition

A solution of the main metricity condition is

$$\Lambda = -a\omega + y^{3/2}(\rho + au) = -a\omega + y^{3/2}(x)$$

for some constant a and where y is given implicitly as $x = \pm\int \frac{y\,dy}{\sqrt{y\left(2ky^2 - Ay - 4a\right)}}$.

- The secondary metricity condition is

$$3\partial\Omega = \Omega\,\partial_\rho \Lambda.$$

- Assuming that $\Omega = \Omega(\rho + au)$ it is easy to show that

$$\Omega = h^{1/3} = y^{1/2}$$

satisfies the secondary metricity condition.

Family of Solutions of the NSF Field Equations

We have found a family of solutions of the NSF field equations given by

$\Lambda = -a\omega + y^{3/2}(\rho + au) = -a\omega + y^{3/2}(x)$ and $\Omega = h^{1/3} = y^{1/2}$

where y is given implicitly as $x = \rho + au = \pm\int \frac{ydy}{\sqrt{y(2ky^2 - Ay - 4a)}}$ for constants, a, k and A.

- Once $\Lambda = \partial^3 Z = \partial^3 u = \partial^2 \omega = \partial\rho = \frac{\partial\rho}{\partial\varphi}$ is found, it can in principle be integrated (three times) to find the families of null surfaces given by:

$$u = Z = \text{constant.}$$

Finding the Null Surfaces

We have found a family of solutions of the NSF field equations given by

$$\Lambda = -a\omega + y^{3/2}(\rho + au) = -a\omega + y^{3/2}(x) \text{ and } \Omega = h^{1/3} = y^{1/2}$$

where y is given implicitly as $x = \rho + au = \pm\int \frac{ydy}{\sqrt{y(2ky^2 - Ay - 4a)}}$ for constants, a, k and A.

- For the special case $a = k = 0, A = -1$ one has $x = \rho = y$ and so

$$\Lambda = \partial^3 Z = \frac{\partial\rho}{\partial\varphi} = \rho^{3/2}$$

can be integrated three times with respect to φ to give

$$\boxed{Z = -4\ln(\varphi + x) + y\varphi + z}$$ where x, y and z are constants of integration.

For example: Recall φ is a parameter that labels the surfaces and so selecting $\varphi = 1$

$$Z = -4\ln(1 + x) + y + z \ = \text{constant}$$ which clearly represents a (null) surface.

Solution of Third NSF Equation

The third NSF equation: $\partial_\rho^2 \Omega = \kappa T_{\rho\rho} \Omega$, with $\Omega = y^{1/2}$ gives

$$T_{\rho\rho} = \frac{1}{4\kappa y^2}\left(A + 8ay^{-1}\right) = \frac{1}{4\kappa}\left(A + 8ay^{-1}\right) y^{-2}$$

The components G_{ij} of the $2+1$ Einstein equations can be found because the associated GR metric (hence R_{ij} and R) can be always be written in terms of Λ and Ω:

$$\left[g_{ij}\right] = \Omega^{-2}\left[\gamma_{ij}\right] = \Omega^{-2} \begin{pmatrix} -\frac{1}{3}\partial\left(\partial_\rho \Lambda\right) + \frac{2}{9}\left(\partial_\rho \Lambda\right)^2 + \partial_\omega \Lambda & \frac{1}{3}\left(\partial_\rho \Lambda\right) & -1 \\ \frac{1}{3}\left(\partial_\rho \Lambda\right) & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Using our choice for Ω and our expression for Λ in terms of y show that in particular:

$$G_{\rho\rho} = R_{\rho\rho} - \frac{1}{2} R g_{\rho\rho} = \frac{1}{4}\left(A + 8ay^{-1}\right) y^{-2}$$

These values are consistent and so the 2+1 Einstein equations $G_{ij} = \kappa T_{ij}$ hold.

Calculating the other T_{ij} components shows these solutions have an imperfect fluid source.

Conformal Flatness: Cotton-York Tensor

NSF does not distinguish between conformally related spacetimes and so, to be interesting, a solution must not be conformally flat.

In 2+1 dimensions, conformal flatness can be tested by the Cotton-York tensor [(16)–(18)]

$$C^i_j = \varepsilon^{ikp}\left(R_{pj} - \tfrac{1}{4} R g_{pj}\right)_{;k}$$

C^i_j is identically zero if and only if the 2+1 spacetime is conformally flat.

(16) Cotton, E.: Ann. Fac. Sci. Toulouse (Sér.2) **1**, 385 (1899)
(17) York Jr., J.W.: Phys Rev. Lett. **26**,1656 (1971)
(18) Garcia-Diaz, A.A.: Exact Solutions in Three-Dimensional Gravity. Cambridge University Press, Cambridge (2017)

Cotton-York Components

The components of the Cotton-York tensor are defined by

$$C^i_j = \varepsilon^{ikp}\left(R_{pj} - \tfrac{1}{4}Rg_{pj}\right)_{;k}$$

and for this family of solutions are

$$\left[C^i{}_j\right] = \begin{pmatrix} -\tfrac{1}{4}AR - \tfrac{1}{128}AW^2 - 2aRy^{-1} & y^{-1/2}\left(\tfrac{1}{2}R + \tfrac{1}{32}A^2\right)\partial_\rho y & -y^{-1}\left(R + \tfrac{1}{16}AW\right) \\ ay^{-1/2}\left(\tfrac{1}{2}R + \tfrac{1}{32}A^2\right)\partial_\rho y & \tfrac{1}{128}A^2W + RAy^{-1} & \tfrac{1}{4}Aay^{-3/2}\partial_\rho y \\ \tfrac{1}{128}AW^2a - 2Ra^2y^{-1} + \tfrac{1}{2}RWa & -ay^{-1/2}\left(\tfrac{1}{2}R + \tfrac{1}{32}A^2\right)\partial_\rho y & \tfrac{1}{16}AWay^{-1} + \tfrac{1}{4}RA + Ray^{-1} \end{pmatrix}$$

where $W = A + 8ay^{-1}$ and the Ricci scalar $R = \dfrac{1}{32}A^2 + ka$.

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where $W = A + 8ay^{-1}$ and the Ricci scalar $R = \dfrac{1}{32}A^2 + ka.$

In general $\left[C^i{}_j\right] \neq 0$ and so members of this family of solutions are not conformally flat.

The only exception is the special circumstance when $A = 0$ and either a or k is also zero, which makes $R = 0$.

Petrov Types

Following Garcia-Diaz[18] the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the Cotton-York tensor $C^{i}{}_{j}$ and a matrix $Q = \left[Q^{i}{}_{j}\right]$ where $Q^i_j = \left(C^{i}{}_{k} - \lambda_3 \delta^i_k\right)\left(C^{k}{}_{j} + \frac{1}{2}\lambda_3 \delta^{k}{}_{j}\right)$ can be used to define a Petrov type classification of a 2+1 spacetime:

Petrov Type I: $\lambda_1 \neq \lambda_2 \neq \lambda_3 = -\lambda_1 - \lambda_2$ eigenvalues real

Petrov Type I′: $\lambda_1 \neq \lambda_2 \neq \lambda_3 = -\lambda_1 - \lambda_2$ two eigenvalues complex, one real: $\lambda_1 = \bar{\lambda}_2;\ \lambda_3 = -2\,\mathrm{Re}\left(\lambda_1\right)$

Petrov Type D: $\lambda_1 = \lambda_2 \neq \lambda_3 = -\lambda_1 - \lambda_2$ and $\left[Q^i_j\right] = 0$

Petrov Type II: $\lambda_1 = \lambda_2 \neq \lambda_3 = -\lambda_1 - \lambda_2$ and $\left[Q^i_j\right] \neq 0$

Petrov Type O: $\lambda_1 = \lambda_2 = \lambda_3 = 0;$ $\left[C^{i}{}_{k}\right] = 0$ conformally flat (trivial case)

Petrov Type N: $\lambda_1 = \lambda_2 = \lambda_3 = 0;$ $\left[C^{i}{}_{k}\right] \neq 0,\ \left[C^{i}{}_{j} C^{j}{}_{k}\right] = 0$

Petrov Type III: $\lambda_1 = \lambda_2 = \lambda_3 = 0;$ $\left[C^{i}{}_{k}\right] \neq 0,\ \left[C^{i}{}_{j} C^{j}{}_{k}\right] \neq 0$

(18) Garcia-Diaz, A.A.: Exact Solutions in Three-Dimensional Gravity. Cambridge University Press, Cambridge (2017)

Examples of Petrov Types

The eigenvalues for our family of solutions are:

$$\left(R = \frac{1}{32} A^2 + ka \right)$$

and include four of the six non-trivial Petrov types.

$$\lambda_1 = \frac{1}{4} AR$$

$$\lambda_2 = -\frac{1}{8} AR + \sqrt{\frac{R}{2}} \left(\frac{1}{16} A^2 + R \right)$$

$$\lambda_3 = -\frac{1}{8} AR - \sqrt{\frac{R}{2}} \left(\frac{1}{16} A^2 + R \right)$$

For general values of the constants a, k and A, the eigenvalues are different

- ▷ if $R > 0$ the eigenvalues all real – Petrov Type I
- ▷ if $R < 0$ $\left(\text{if } ka < 0 \text{ and } |ka| > A^2/32\right)$, eigenvalues are complex – Petrov Type I′

Certain specific values of the constants a, k and A generate other Petrov types:

- ▷ When $a = 0$ and $A \neq 0$: $\lambda_1 = \lambda_2 = A^3/128$, $Q^i{}_j = 0$ – Petrov Type D
- ▷ When $k = 0$, but $a \neq 0$ and $A \neq 0$: $\lambda_1 = \lambda_2 = A^3/128$, $Q^i{}_j \neq 0$ – Petrov Type II

Conclusions

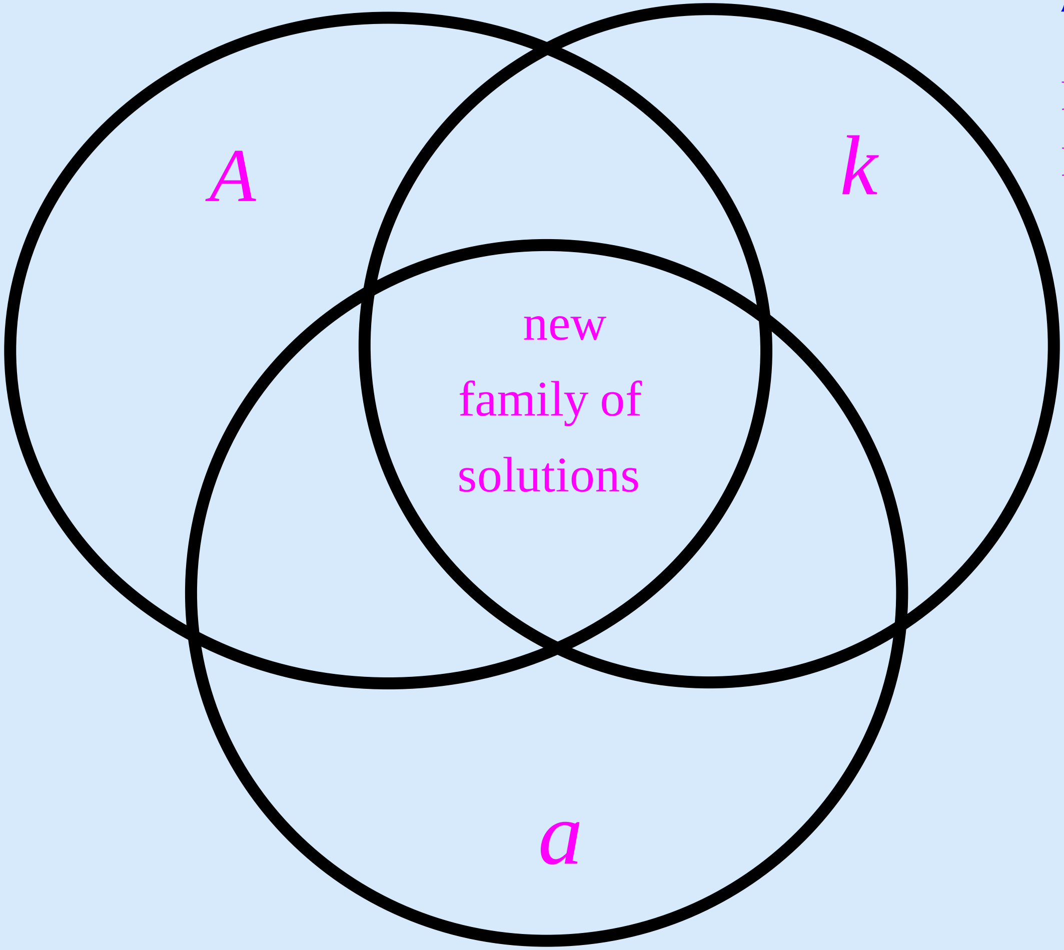
The solution presented here $ds^2 = -\left(\frac{1}{4}A + 3ay^{-1}\right)du^2 + y^{-1/2}\left(\partial_\rho y\right)du\,d\omega - 2y^{-1}du\,d\rho + d\omega^2$,

$\Lambda\left(u,\omega,\rho\right) = -a\omega + h\left(\rho + au\right) = -a\omega + h\left(x\right)$ is just the fourth known solution of the 2+1 NSF.

$$\rho + au = \pm\int \frac{y\,dy}{\sqrt{y(2ky^2 - Ay - 4a)}}$$

Family of solutions

Imperfect fluid source



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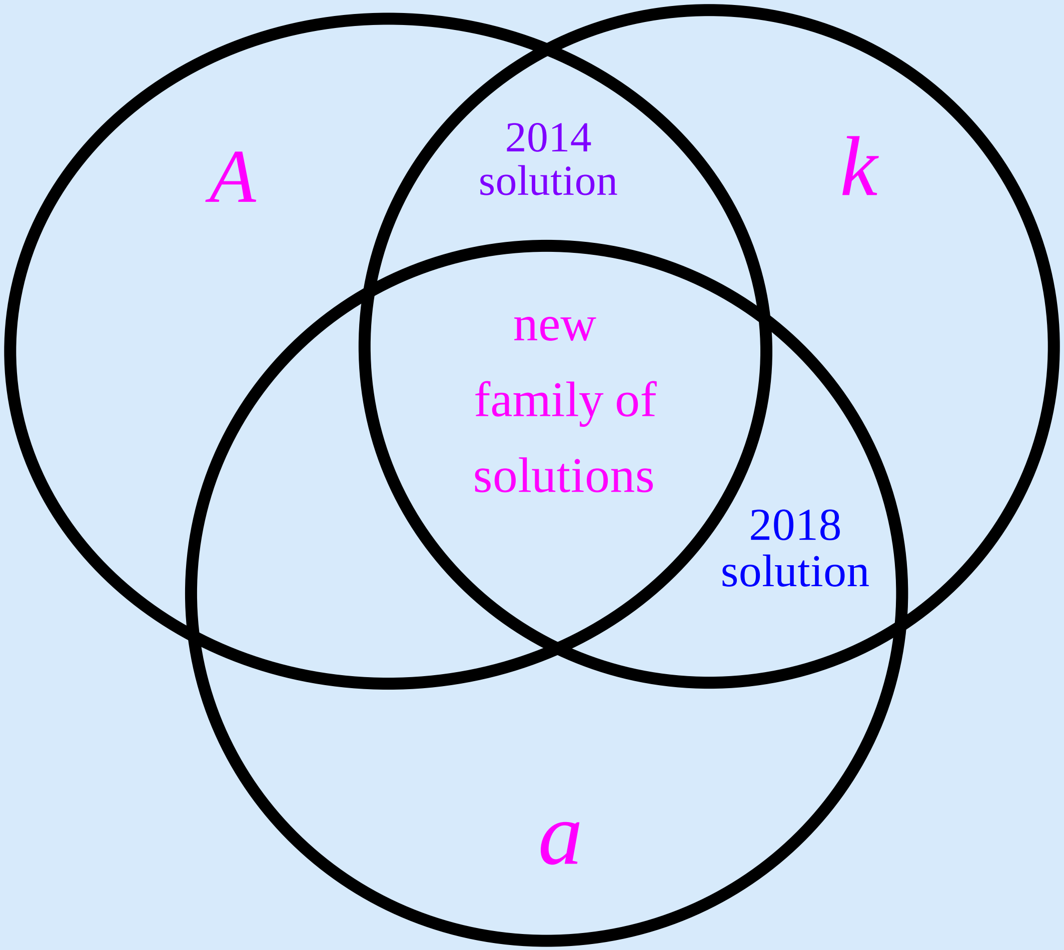

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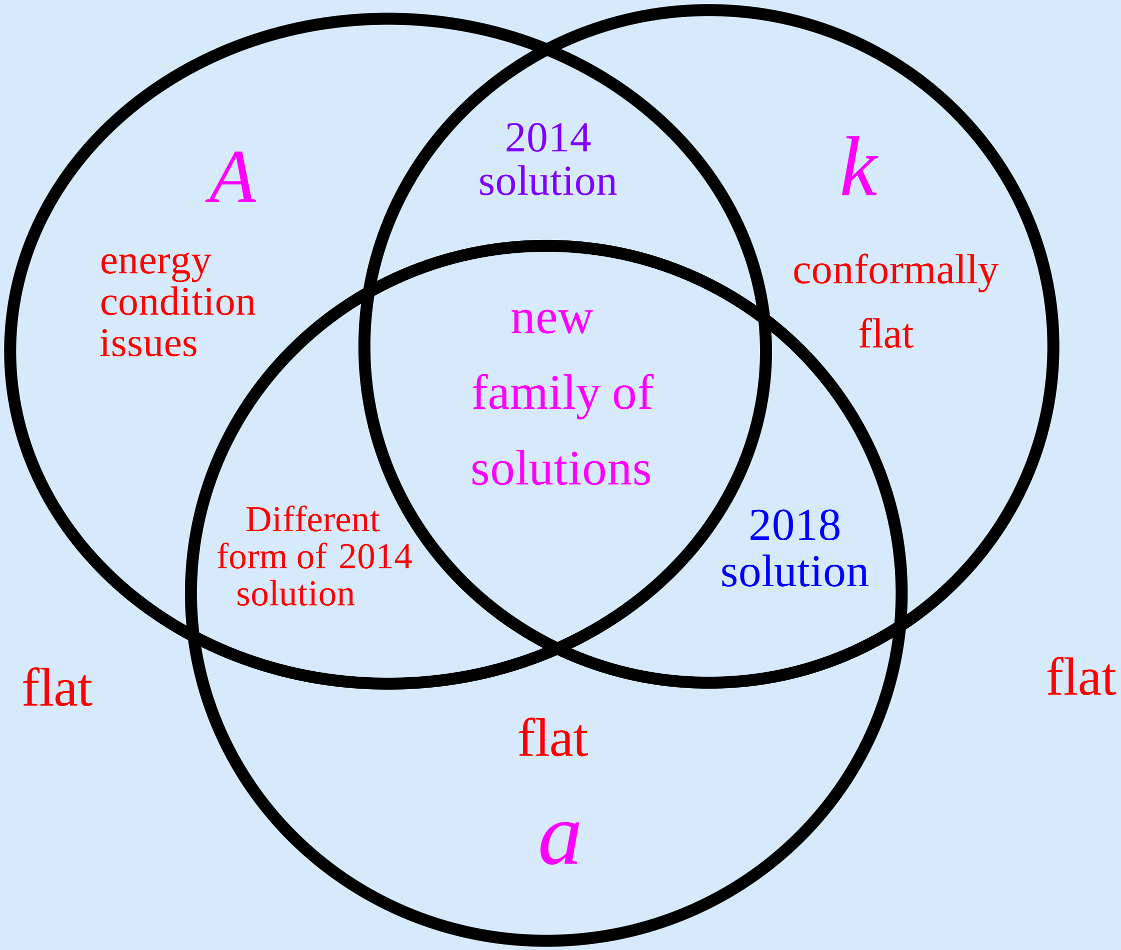

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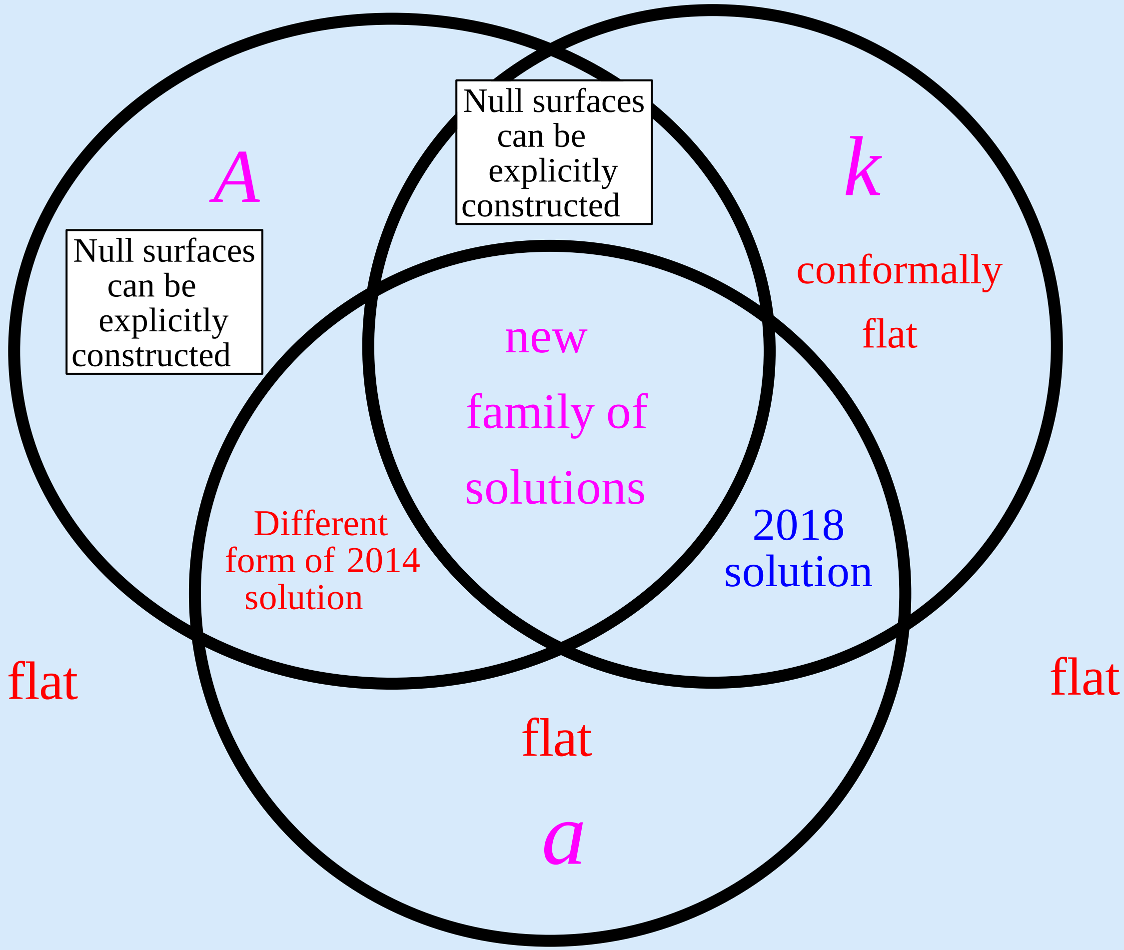

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