Cosmological static metrics and Kruskal coordinates from symmetry transformations

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In short

- We derive the static form for the FLRW model from metric transformations.

- Extend the formalism to obtain a new view of Kruskal coordinates.

- Obtain Kruskal for any given spherically symmetric static metric

- Obtain some known transformations and two novel explicit transformations.

J.A. Nieto, E. A. León, C. García-Quintero, Rev. Mex. Fis. 68 (2022) 4, 040701

Early relativity

In the very first models, the role of coordinates was somewhat "messy". For instance:

- de Sitter Universe (static or evolving with time?)

- Schwarzschild spacetime in several coordinates:

  - Schwarzschild
  - Lemaître
  - Painleve-Gullstrand
  - Isotropic
  - Eddington-Finkelstein
  - Kruskal-Szekeres!

A. Mitra, Nature Sci. Rep., 2012

From cosmological evolving metrics to “static” ones

Start with Einstein’s equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 8\pi G T_{\mu\nu}.$$

FLRW metric solution

$$dS_{(1)}^2 = -dT^2 + a^2(T)\left(\frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\Omega^2\right)$$

Satisfying

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho_f + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_f + 3p_f) + \frac{\Lambda}{3}.$$

Consider transform this metric to the form

$$dS^2_{(2)} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2.$$

T. Jacobson, Class. Quant. When is gttgrr = −1?, Grav. (2007).

$$r = a\rho.$$

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\gamma_{\alpha\beta}$$

(tensor transformation)

We obtain some relations

$$\dot{t}^2 = \frac{1}{f^2}(f + \rho^2 \dot{a}^2),$$

$$t'^2 = \frac{a^2}{f^2}\left(1 - \frac{f}{1 - k\rho^2}\right)$$

$$f^4 \dot{t}^2 t'^2 = \rho^2 a^2 \dot{a}^2$$

$$f = 1 - \rho^2(\dot{a}^2 + k)$$

After some manipulation, this appears

$$[(\dot{a}^2 + k) - a\ddot{a}]\,[1 + \rho^2(\dot{a}^2 - k)] = 0$$

Only consistent with:

$$\frac{\ddot{a}}{a} = \frac{\dot{a}^2 + k}{a^2} \quad \Rightarrow \quad \frac{\dot{a}^2 + k}{a^2} = \Gamma$$

(Friedmann equation for vacuum)

Explicit solutions are

Curvature	Metric
$k=0, \Gamma>0$	$dS^2 = -dT^2 + e^{2\sqrt{\Gamma}T}\left(d\rho^2+\rho^2 d\Omega^2\right)$
$k=1, \Gamma>0$	$dS^2 = -dT^2 + \frac{\cosh^2\left(\sqrt{\Gamma}T\right)}{\Gamma}\left(\frac{d\rho^2}{1-\rho^2}+\rho^2 d\Omega^2\right)$
$k=-1, \Gamma>0$	$dS^2 = -dT^2 + \frac{\sinh^2\left(\sqrt{\Gamma}T\right)}{\Gamma}\left(\frac{d\rho^2}{1+\rho^2}+\rho^2 d\Omega^2\right)$
$k=-1, \Gamma<0$	$dS^2 = -dT^2 + \frac{\sin^2\left(\sqrt{\lvert\Gamma\rvert}T\right)}{\lvert\Gamma\rvert}\left(\frac{d\rho^2}{1+\rho^2}+\rho^2 d\Omega^2\right)$

All consistent with (solving for function f):

$$dS^2_{(2)} = -\left(1-\Gamma r^2\right)dt^2 + \frac{dr^2}{1-\Gamma r^2} + r^2 d\Omega^2$$

Florides, Gen. Rel. Grav. (1980). Melia, MNRAS (2012). Mitra, Int. J. Mod. Phys D (2015).

Also, we checked a (bit) more general metric:

$$dS_{(3)}^2 = -dT^2 + a^2(T)\left(b^2(\rho)d\rho^2 + \rho^2 d\Omega^2\right)$$

Similar manipulations lead to

$$b' = \rho b^3\left(a\ddot{a} - \dot{a}^2\right) \quad \Rightarrow \quad a\ddot{a} - \dot{a}^2 = \frac{b'}{\rho b^3} = \kappa$$

(Given dependences of a and b)

Integrating last equality yields FLRW again

$$\frac{1}{b^2} = B - \kappa\rho^2$$

Extending to include Kruskal transformations

Assume metrics that can take a conformally flat form at space-time radial slices:

$$dS^2_{(4)} = N^2(T,\rho)(-dT^2 + d\rho^2) + \varphi^2(T,\rho)d\Omega^2$$

remember $dS^2_{(2)} = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$

The tensor transformation now yields

$$(\partial_t T)^2 = (\partial_t \rho)^2 + f N^{-2}$$

$$(\partial_r T)^2 = (\partial_r \rho)^2 - f^{-1} N^{-2}$$

$$\partial_t T \partial_r T = \partial_t \rho \partial_r \rho.$$

$$N^{-2} = (\partial_r \rho)^2 f - (\partial_t \rho)^2 f^{-1}$$

That allows to simplify relations to

$$\partial_t T = f \partial_r \rho \qquad \partial_r T = f^{-1} \partial_t \rho.$$

By assuming $T(t,r) = \Theta(t)\Phi(r)$ $\rho(t,r) = \xi(t)\chi(r)$

$$\frac{1}{\xi}\frac{d\Theta}{dt} = \frac{f}{\Phi}\frac{d\chi}{dr} = \alpha \qquad \frac{1}{\Theta}\frac{d\xi}{dt} = \frac{f}{\chi}\frac{d\Phi}{dr} = \beta$$

Implying

$$\frac{d^2\Theta}{dt^2} - \alpha\beta\Theta = 0. \qquad \alpha\Phi^2 - \beta\chi^2 = \sigma$$

$Case\ \alpha\beta = 0$

Transformation maps onto itself.

$$N^2 = -f/\left(\partial_t \rho\right)^2 = -f.$$

Special case: For Schwarzschild, Tortoise Coordinates

$$-\left(1 - r_s/r\right)\left(dt^2 + dr^{*2}\right)$$

$Case\ \alpha\beta < 0$

$$T(t,r) = \sin\left(\sqrt{-\alpha\beta}\int \frac{dr}{f}\right)\sin\left(\sqrt{-\alpha\beta}t\right)$$

$$\rho(t,r) = \cos\left(\sqrt{-\alpha\beta}\int \frac{dr}{f}\right)\cos\left(\sqrt{-\alpha\beta}t\right)$$

$$N^2 = \frac{f}{-\alpha\beta\left[\cos\left(2\sqrt{-\alpha\beta}t\right) - \cos\left(2\sqrt{-\alpha\beta}\int \frac{dr}{f}\right)\right]}$$

Not (yet) explored in literature. Interpretation still open.

$$Case\ \alpha\beta > 0$$

Choosing convenient signs, and developing:

$$\frac{1}{\Phi}\frac{d\Phi}{dr} = \pm\sqrt{\alpha\beta}f^{-1}$$

$$\Phi(r) = Ae^{\pm\sqrt{\alpha\beta}\int\frac{dr}{f}}$$

$$\chi(r) = \pm A\sqrt{\frac{\alpha}{\beta}}e^{\pm\sqrt{\alpha\beta}\int\frac{dr}{f}}$$

$$N^2 = \frac{f\Phi^{-2}}{\alpha\beta}$$

$$T(t,r) = \Phi(r)\sinh\left(\sqrt{\alpha\beta}t\right)$$

$$\rho(t,r) = \Phi(r)\cosh\left(\sqrt{\alpha\beta}t\right)$$

Also remember

$$dS^2_{(4)} = N^2(T,\rho)(-dT^2 + d\rho^2) + \varphi^2(T,\rho)d\Omega^2$$

Clearly Kruskal!

Some particular solutions:

Schwarzschild:

$$T_{\rm Schw} = e^{\frac{r}{2r_s}} \sqrt{\frac{r}{r_s} - 1} \sinh \frac{t}{2r_s}$$

$$\rho_{\rm Schw} = e^{\frac{r}{2r_s}} \sqrt{\frac{r}{r_s} - 1} \cosh \frac{t}{2r_s}$$

$$dS^2_{\rm Schw} = \frac{4r_s^3}{r} e^{-\frac{r}{r_s}} (-dT^2 + d\rho^2) + r^2 d\Omega^2.$$

$$\rho^2_{\rm Schw} - T^2_{\rm Schw} = e^{\frac{r}{r_s}} (r/r_s - 1)$$

De-Sitter:

$$dS^2_{\rm dS} = \frac{1}{\Gamma} \left(1 - \sqrt{\Gamma} r\right)^2 (-dT^2 + d\rho^2) + r^2 d\Omega^2$$

$$T_{\rm dS}(t, r) = \frac{1-\sqrt{\Gamma}r}{\sqrt{1-\Gamma r^2}} \sinh\left(\sqrt{\Gamma} t\right)$$

$$\rho_{\rm dS}(t, r) = \frac{1-\sqrt{\Gamma}r}{\sqrt{1-\Gamma r^2}} \cosh\left(\sqrt{\Gamma} t\right)$$

$$\rho^2_{\rm dS} - T^2_{\rm dS} = \left(1 - \sqrt{\Gamma} r\right) / \left(1 + \sqrt{\Gamma} r\right)$$

(both known)

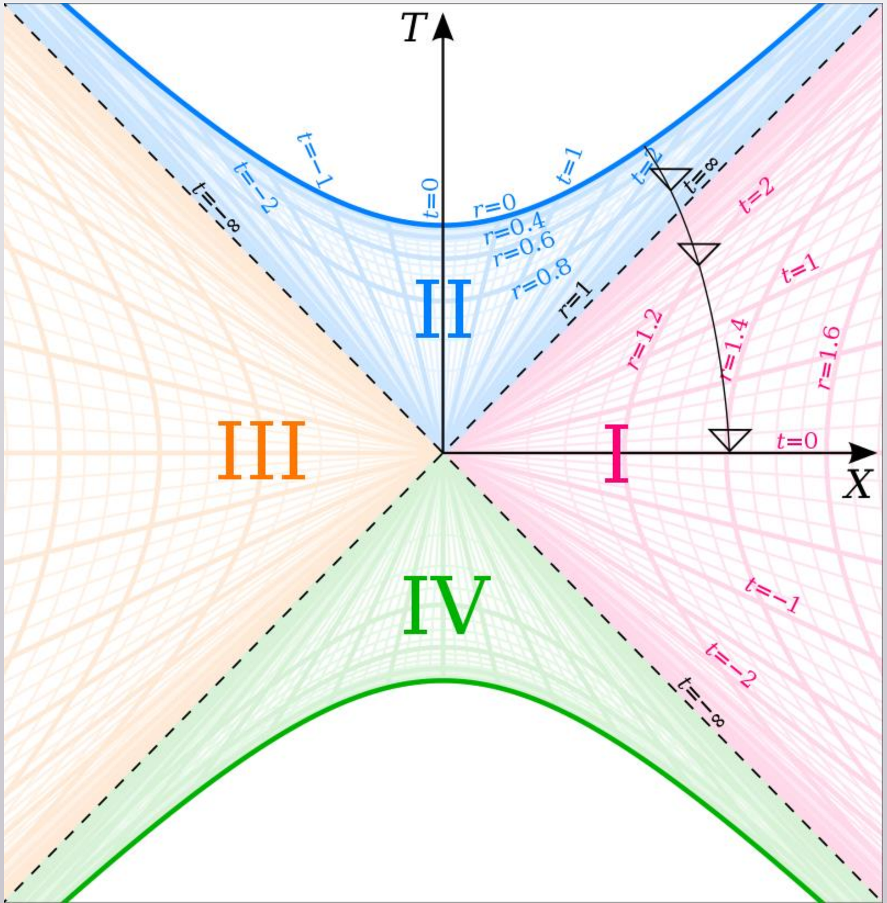
Schwarzschild:

$$T_{\mathrm{Schw}} = e^{\frac{r}{2r_s}} \sqrt{\frac{r}{r_s} - 1}\, \sinh \frac{t}{2r_s}$$

$$\rho_{\mathrm{Schw}} = e^{\frac{r}{2r_s}} \sqrt{\frac{r}{r_s} - 1}\, \cosh \frac{t}{2r_s}$$

$$dS^2_{\mathrm{Schw}} = \frac{4r_s^3}{r} e^{-\frac{r}{r_s}} (-dT^2 + d\rho^2) + r^2 d\Omega^2.$$

$$\rho^2_{\mathrm{Schw}} - T^2_{\mathrm{Schw}} = e^{\frac{r}{r_s}} \left(r/r_s - 1\right)$$



https://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg

Extremal Reissner-Nordström:

$\varphi_1 = (1+\sqrt{5})/2$ and $\varphi_2 = (1-\sqrt{5})/2$

Golden ratios!

$$dS^2_{\text{Ex}} = \frac{16 r_+^4}{r^2}\left(\frac{r}{r_+}-1\right) e^{\frac{-\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{\left(\frac{r}{r_+}-1\right)}} (-dT^2 + d\rho^2) + r^2 d\Omega^2$$

$$T_{\text{Ex}}(t,r) = e^{\frac{\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{4\left(\frac{r}{r_+}-1\right)}} \sqrt{\frac{r}{r_+}-1}\,\sinh\frac{t}{4r_+},$$

$$\rho_{\text{Ex}}(t,r) = e^{\frac{\left(\frac{r}{r_+}-\varphi_1\right)\left(\frac{r}{r_+}-\varphi_2\right)}{4\left(\frac{r}{r_+}-1\right)}} \sqrt{\frac{r}{r_+}-1}\,\cosh\frac{t}{4r_+}$$

$$\rho^2_{\text{Ex}} - T^2_{\text{Ex}} = (r/r_+ - 1)\exp\{(r/r_+ - \varphi_1)(r/r_+ - \varphi_2)/[2(r/r_+ - 1)]\}$$

Also solved for Schwarzschild-de Sitter… (both) not found before

A generalized de-Sitter metric: $f = 1 - h^2 r^2 + q^4 r^4$

$$r_{\pm}^2 = (h^2 \pm \sqrt{h^4 - 4q^4})/(2q^4)$$

$$T_{\mathrm{GdS}}(t,r) = \sqrt{\frac{r_+ - r}{r_+ + r}} e^{-\frac{r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)} \sinh(\gamma t)$$

$$\gamma = -q^4(r_+^2 - r_-^2) r_+$$

$$\rho_{\mathrm{GdS}}(t,r) = \sqrt{\frac{r_+ - r}{r_+ + r}} e^{-\frac{r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)} \cosh(\gamma t)$$

$$\rho_{\mathrm{GdS}}^2 - T_{\mathrm{GdS}}^2 = \frac{r_+ - r}{r_+ + r} e^{-\frac{2r_+}{r_-}\tan^{-1}\left(\frac{r}{r_-}\right)}$$

P.F. Gonzalez-Diaz, Phys. Rev. D (2000).

Final comments

1. We derived cosmological transformation to static form. Showed (cleaner way) uniqueness of solutions.

2. For conformally flat transformations, we obtained several possibilities:
   - A rather trivial one, where prototype is tortoise coordinates.
   - An interesting one with usual sines and cosines in place of usual hyperbolic functions.
   - One that yields the Kruskal-type transformation.

3. For the later, we obtained solutions that are consistent with those found in books and articles. But also we obtained some new solutions.

Thank you for attending!