Fefferman-Graham obstruction tensor and Einstein's equations

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Conformal method of Friedrich

Penrose compactification in terms of an unphysical conformally rescaled metric

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega \in C^\infty(M)$$

- Einstein's equations $G_{\mu\nu}[g] = \Lambda g_{\mu\nu}$, written in terms of $\hat{g}_{\mu\nu}$ are singular at conformal boundary surface $\Omega = 0$.
- Not directly applicable to study asymptotics.

Conformal method of Friedrich

Friedrich's approach

Find system of equations which

1. are more general i.e. every solution to Einstein's equations is also a solution to these equations,
2. are conformally invariant,
3. the system is hyperbolic (after imposing suitable gauge),
4. the scale factor Ω and properties of being conformal to Einsteinian metric propagate by hyperbolic equations too.

- Stability of asymptotically simple spaces (possesing Penrose compactification) follows from **well-posedness** of hyperbolic equations.
- Smoothness of Penrose compactification $(\Lambda > 0)$.

Friedrich's solution invented for $3 + 1$.

Anderson's proposition (dimension $d \geq 4$ even)

Vanishing of Fefferman-Graham obstruction tensor $H_{\mu\nu}$ follows from Einstein's equations

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \implies H_{\mu\nu} = 0$$

Use equation $H_{\mu\nu} = 0$ (AFG equations).

- Nice conformal transformations of $H_{\mu\nu}$ (by powers of Ω)
- Lagrangean formulation, $\nabla^\mu H_{\mu\nu} = 0$ and $H^\mu_\mu = 0$

Complicated high order tensor

$$H_{\mu\nu} = c\Box^{d/2-2} B_{\mu\nu} + \ldots, \quad c \in \mathbb{R}$$

with the Bach tensor $B_{\mu\nu} = \Box P_{\mu\nu} - \nabla^\chi \nabla_\mu P_{\chi\nu} + \ldots$, where $P_{\mu\nu}$ is the Schouten tensor .

Is this eqaution well-posed after gauge fixing?

In $d = 4$ well-posedness proved by Guenther '70. Proofs in higher dimensions nontrivial (first approach Anderson, Anderson-Chruściel).

Anderson proposition (AFG equations)

In order to obtain well-defined evolution we need to impose the gauge conditions (similarly as for the Einstein's equations)

1. Gauge freedom: diffeomorphisms and conformal transformations

$$F_\mu = \Box x_\mu = 0, \quad R = 0 \quad \text{(gauge fixing, always possible)}$$

2. Constraints $H_{\mu\nu} n^\mu|_\Sigma = 0$ for initial data $D^{d-1} g_{\mu\nu}|_\Sigma$

The gauge fixed equation is now

$$\boxdot^{\frac{d}{2}} g_{\alpha\beta} + F(D^{d-1} g) = 0, \quad \boxdot = g^{\mu\nu}\partial_\mu\partial_\nu$$

Weakly hyperbolic (we can compute all time derivatives, convergent series for analytic initial data).

Weakly hyperbolic systems

Equations:

$$\Box^{\frac{d}{2}} g_{\alpha\beta} + F(D^{d-1}g) = 0, \quad \Box = g^{\mu\nu}\partial_\mu\partial_\nu$$

1. Multiple characteristics of the principal symbol $(p_\mu p^\mu)^{d/2}$ of linearization. In general such equations are not well-posed.
2. Problems already for linear equations with constant coefficients (multiple roots in characteristic polynomial can aquire large imaginary part via lower order perturbation) that leads to arbitrary fastly growing modes.
3. Some additional properties of lower order terms are necessary (Levi-like conditions). These are difficult to check for the gauge fixed FG obstruction tensor.

Contrast with Einstein's equations in harmonic gauge (quasi-linear wave equation, stable hyperbolic, well-posed).

Ambient metric equations

The AFG is not a random equation, but it is related to Einstein's equations in the ambient metric:

Ambient metric construction of Fefferman-Graham

Associate $d+2$ dimensional metric with a conformal structure $[g_{\mu\nu}]$

$$\mathbf{g} = 2\rho dt^2 + 2t dt d\rho + t^2 \tilde{g}_{\mu\nu}(x^\mu, \rho) dx^\mu dx^\nu, \quad \mathbf{T} = t\partial_t \text{ conf. Killing}$$

where $\tilde{g}_{\mu\nu} = \sum_{n=0} \tilde{g}^{[n]}_{\mu\nu} \rho^n$ is a ρ-dependent metric on M ($\tilde{g}^{[0]}_{\mu\nu} = g_{\mu\nu}$).

The wave equation in the ambient metric induces well-behaved hyperbolic high order equation. Example:

(Critical) Graham-Jenne-Mason-Sparling (GJMS) equation

$$0 = P_d \phi = \boxdot^{d/2} \phi + \ldots \iff \Box \boldsymbol{\phi} = O(\rho^{d/2}), \quad \mathcal{L}_{\mathbf{T}} \boldsymbol{\phi} = 0$$

where $\phi = \boldsymbol{\phi}|_{\rho=0, t=1}$.

Well-posedness

Equivalent formula for $\phi|_{t=1} = \tilde{\phi} = \sum_{n=0}^{d/2-1} \tilde{\phi}^{[n]} \rho^n + \dots$

$$[\Box_{\tilde{g}} \tilde{\phi}]^{[n]} - 2\left(n + 1 - \frac{d}{2}\right)(n+1)\tilde{\phi}^{[n+1]} = 0.$$

Decoupled system of equations for $\tilde{\phi}^{[n]}$ ($n \leq \frac{d}{2} - 1$), scalar fields on M.

Recursive systems

We can recursively determined higher expansion fields and the system is equivalent to high order equation for $\tilde{\phi}^{[0]}$.

Generalized quasilinear wave equation is well-posed

The system

$$[\Box_{\tilde{g}(\tilde{u})} \tilde{u}]^{[n]} + F^n(D^1 \tilde{u}^{[k]}_{k \leq n}, \tilde{u}^{[n+1]}) = 0, \quad 0 \leq n \leq N.$$

for $\tilde{u} + O(\rho^{N+1})$. Here $\tilde{g}(\tilde{u})$ means that $\tilde{g}^{[n]}$ depends on $\tilde{u}^{[k]}_{k \leq n}$.

Function space: shifted Sobolev spaces.

Example for GJMS in $d = 4$

System of equations for two scalar fields on M

$$\tilde{\phi}^{[0]}, \quad \tilde{\phi}^{[1]}.$$

Expansion $\Box_{\tilde{g}} = \Box + \rho\Box^{[1]}$, where $\Box^{[1]} = f^{\mu\nu}\partial_\mu\partial_\nu + f^\mu\partial_\mu$

$$\begin{cases} \Box\tilde{\phi}^{[0]} + 2\tilde{\phi}^{[1]} = 0 \\ \Box\tilde{\phi}^{[1]} + \Box^{[1]}\tilde{\phi}^{[0]} = 0 \end{cases} \Longrightarrow \begin{cases} \Box\tilde{\phi}^{[0]} + 2\tilde{\phi}^{[1]} = 0 \\ \Box\tilde{\phi}^{[1]} + f^{\mu\nu}\partial_\mu p_\nu + f^\mu\partial_\mu\tilde{\phi}^{[0]} = 0 \\ \Box p_\mu + 2\partial_\mu\tilde{\phi}^{[1]} + \ldots = 0 \end{cases}$$

after introducing dependent variables $p_\mu = \partial_\mu\tilde{\phi}^{[0]}$.
Generalized to higher orders by induction.

Excessive gauge fixing.

Obstruction tensor

$$H_{\mu\nu} = 0 \Longleftrightarrow \mathbf{R}_{\mu\nu} = O_+(\rho^{d/2}) \tag{1}$$

Recursively compute $\tilde{g}^{[n]}_{\mu\nu}$ and plug to the last equation for $g_{\mu\nu} = \tilde{g}^{[0]}_{\mu\nu}$.
Should be treated as a second order evolution system

Problems with the method of Choquet-Bruhat

1. The propagation of the gauge in Choquet-Bruhat method uses Bianchi identity. Here Bianchi identity are used to recover

$$\mathbf{R}_{\mu\infty} = O(\rho^{d/2-1}) \text{ and } \mathbf{R}_{\infty\infty} = O(\rho^{d/2-2})$$

   from $\mathbf{R}_{\mu\nu} = O_+(\rho^{d/2})$ (∞ is ρ direction).
2. We construct gauge fixing functions $\tilde{\gamma}$ and $\tilde{G}_\mu$ from $\mathbf{R}_{\mu\infty}$ and $\mathbf{R}_{\infty\infty}$.

Choquet-Bruhat method for AFG

1. Introduce gauge fixing functions (notation $\tilde{S}_{IJ} = \mathbf{R}_{IJ}|_{t=1}$).

$$\tilde{\gamma} = -\frac{1}{2}\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi} + \partial_\infty^{-1}\tilde{S}_{\infty\infty}, \quad \tilde{G}_\mu = F_\mu + 2\partial_\infty^{-1}\tilde{S}_{\mu\infty} - \partial_\mu\partial_\infty^{-1}\tilde{\gamma},$$

2. The gauge fixed equation $\tilde{E}_{\mu\nu} = O(\rho^{d/2})$ for $\tilde{g}_{\mu\nu} + O(\rho^{d/2})$.

$$\tilde{E}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu\tilde{G}_\nu + \tilde{\nabla}_\nu\tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma}.$$

   is decoupled, recursive and generalized hyperbolic.
3. Bianchi identity gives decoupled, recursive and generalized hyperbolic equations for the gauge functions $\tilde{\gamma} + O(\rho^{d/2-1})$ and $\tilde{G}_\mu + O(\rho^{d/2})$.
4. Vanishing of the initial condition for this system follows from vanishing of $\tilde{\gamma}^{[0]} \propto R$ and $\tilde{G}^{[0]}_\mu = F_\mu = \Box x_\mu$ up to a sufficient order on the Cauchy surface and constraints there.
5. ... plus standard gluing argument.

AFG equation is well-posed.

Propagation of (almost) Einstein structure

Conformally almost Einstein (Nurowski-Gover, Gover, Graham-Willse)

Existence of the covariantly constant covector

$$\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1})$$

We have $\Omega = \mathbf{I}_0|_{t=1,\rho=0}$ and $\mathbf{I}_I \mathbf{I}^I \propto \Lambda + O(\rho^{d/2-1})$.

- Analogous to propagation of Killing equation for vacuum solutions, but here additionally $\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$ (for $\mathcal{L}_{\mathbf{T}} \boldsymbol{\sigma} = \boldsymbol{\sigma}$)
- If $\Box \boldsymbol{\sigma} = O(\rho^{d/2+1})$ and $\mathbf{R}_{IJ} = O(\rho^\infty)$ (flat extension possible) then

$$(\Box + \ldots)\, \boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1}) \text{ both eqs. well-posed.}$$

- The vanishing of initial data for $\boldsymbol{\nabla}_I \mathbf{I}_J|_{t=1}^{[n]}$ reduces to the standard condition

$$D^{d-1}\, \mathrm{tf}(\nabla_\mu \nabla_\nu \Omega - P_{\mu\nu}\Omega)|_\Sigma = 0$$

thanks to recursive structure of the propagation equation.

Summary

1. The AFG equation (vanishing of the Fefferman-Graham obstruction tensor) is a well-posed system (in $\Box x_\mu = 0$ and $R = 0$ gauge).
2. The almost Einstein condition propagates by hyperbolic equation too, thus we have stability of future or past asymptotically simple solutions (Anderson, Anderson-Chruściel).
3. Application to other equations constructed by ambient metric like conformal powers of d'Alembertians (GJMS), Q-curvature etc.