Hidden symmetries in the geodesic motion from disformal transformations: the pp-wave case in Lorentz and Finsler geometries

Nikolaos Dimakis

Sichuan University

July 5, 2022

N.D., "Hidden symmetries from distortions of the conformal structure"
arXiv preprint: 2205.14973 [gr-qc]

Outline

The general geodesic system

The Lagrangian in the einbein formalism

An equivalent Lagrangian to $L_{sq} = -m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$

$$L = \frac{1}{2N} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - N\frac{m^2}{2} \tag{1}$$

$x = x(\tau)$ the trajectory of the particle, $N(\tau)$ the einbein field, τ the parameter along the curve
(units $c = 1$)

Equations of motion

$$\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa \dot{x}^\lambda = \dot{x}^\mu \frac{d}{d\tau}(\ln N) \tag{2a}$$

$$\frac{1}{N^2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2 = 0 \tag{2b}$$

null geodesics: $m = 0$, (2b)$\Rightarrow g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$

Hamiltonian in the einbein formalism

Total Hamiltonian from the Dirac-Bergmann algorithm

$$H_T = \frac{N}{2}\mathcal{H} + u_N p_N, \tag{3}$$

with the first class constraints

$$p_N = \frac{\partial L}{\partial \dot{N}} = 0 \tag{4}$$

$$\mathcal{H} = g^{\mu\nu} p_\mu p_\nu + m^2 = 0 \tag{5}$$

where $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{N} g_{\mu\nu} \dot{x}^\nu$

$\mathcal{H} = 0$ is the Hamiltonian constraint

Symmetries of L

Point symmetries, generators of the form $Y = Y^\mu(x)\frac{\partial}{\partial x^\mu}$

$m \neq 0$	$m = 0$
KVs	CKVs
$\mathcal{L}_Y g_{\mu\nu} = 0$	$\mathcal{L}_Y g_{\mu\nu} = 2\omega(x) g_{\mu\nu}$

Conserved charges: $I = Y^\mu \frac{\partial L}{\partial \dot{x}^\mu} = Y^\mu p_\mu$

Explicit symmetry breaking when $m \neq 0$

$$\text{KVs} \subset \text{CKVs}$$

What happens with the broken symmetries when $m \neq 0$?

Nonlocal conserved charge [N.D., P. Terzis, T. Christodoulakis, PRD **99**, 104061 (2019)]

$$I = Y^\mu p_\mu + m^2 \int N(\tau)\omega(x(\tau))d\tau \quad , \quad \mathcal{L}_Y g_{\mu\nu} = 2\omega(x) g_{\mu\nu} \tag{6}$$

Generated by

$$X = \left(Y^\mu(x) - \frac{\dot{x}^\mu}{N} \int N\omega(x)d\tau \right) \frac{\partial}{\partial x^\mu} \tag{7}$$

Conservation law:

$$\frac{dI}{d\tau} = \frac{\partial I}{\partial \tau} + \{Y^\mu p_\mu, H_T\} = N\omega(x)\mathcal{H} = 0 \tag{8}$$

- If Y is a Killing vector, $\omega = 0$

$$I = Y^\mu p_\mu$$

- If the particle is massless, $m = 0$

$$I = Y^\mu p_\mu \quad , \quad \mathcal{L}_Y g_{\mu\nu} = 2\omega(x) g_{\mu\nu}$$

The pp-wave case

In Brinkmann coordinates $x^\mu = (v, u, \mathrm{x}, \mathrm{y})$

$$ds^2 = H(u, \mathrm{x}, \mathrm{y})du^2 + 2dudv + \delta_{ij}dx^i dx^j, \quad x^i = (\mathrm{x}, \mathrm{y}) \tag{9}$$

components of CKVs: $Y = Y^\mu \partial_\mu$

$$Y^u = \frac{\mu}{2}\delta_{ij}x^i x^j + a_i(u)x^i + a(u) \tag{10a}$$

$$Y^v = -\mu v^2 + \Big(x^i a_i'(u) + 2b(u) - a'(u)\Big) v + M(u, x, y) \tag{10b}$$

$$Y^i = -\Big(\mu x^i + a_i(u)\Big) v + \gamma_{ijkl}a_j'(u)x^k x^l + b(u)x^i - \epsilon_{ij}c(u)x^j + c_i(u), \tag{10c}$$

$M(u, \mathrm{x}, \mathrm{y})$ satisfies certain integrability conditions and $H(u, \mathrm{x}, \mathrm{y})$

$$\Big[\mu x^i + a_i(u)\Big] \partial_i H = 2\mu H + 2a_i''(u)x^i - 2a''(u) + 4b'(u). \tag{11}$$

Conformal factor

$$\omega(u, v, \mathrm{x}, \mathrm{y}) = b(u) + x^i a_i'(u) - \mu v\,. \tag{12}$$

From nonlocal to local

[M. Elbistan, N.D., K. Andrzejewski, P. A. Horvathy, P. Kosiński, P.-M. Zhang, *Annals Phys.* **418**, 168180 (2020)]

Integrals of motion

$$I = Y^{\mu} p_{\mu} + m^2 \int N(\tau)\omega(x(\tau))d\tau$$

Due to the KV: $\ell = \frac{\partial}{\partial v}$, we have $\frac{dp_v}{d\tau} = 0 \Rightarrow p_v = const. = \pi_v$

$$\ddot{u} - \frac{\dot{N}}{N}\dot{u} = 0 \Rightarrow N = \frac{\dot{u}}{\pi_v} \tag{13}$$

On mass shell we can write

$$m^2 \int N(\tau)\omega(x(\tau))d\tau = \frac{m^2}{\pi_v^2} f^{\alpha} p_{\alpha}$$

$$
\begin{aligned}
f^u =& 0, \\
f^v =& \frac{1}{2} u \left(x^i a_i'(u) - a'(u) + 2b(u) - 2\mu v \right) + \frac{1}{2} x^i a_i(u) + \frac{\mu}{4} \delta_{ij} x^i x^j \\
& + \frac{1}{2} a(u) - \frac{m^2}{\pi_v^2} \frac{\mu}{4} u^2 , \\
f^i =& -\frac{1}{2} u \left(\mu\, x^i + a_i(u) \right) .
\end{aligned}
$$

Final expression for the integrals of motion:

$$
I = \left(Y^\mu + \frac{m^2}{\pi_v^2} f^\mu \right) p_\mu \tag{14}
$$

Conservation law:

$$
\frac{dI}{d\tau} = -2N\omega_m E_N(L) - X^\mu E_\mu(L) - \frac{m^2}{N} \omega_m \left(N^2 - \frac{\dot{u}^2}{\pi_v^2} \right) \tag{15}
$$

where $\omega_m = \omega - \frac{m^2}{2\pi_v^2} \mu\, u$.

Conserved charge from the distorted CKV

$$X^\mu = Y^\mu + \frac{m^2}{\pi_v^2} f^\mu \tag{16}$$

Actual Noether Symmetry: $\mathcal{H} = 0 \Rightarrow m^2 = -g^{\mu\nu} p_\mu p_\nu$
and $\pi_v^2 = p_v^2 = K^{\mu\nu} p_\mu p_\nu$, where $K = \partial_v \otimes \partial_v$

$$\tilde{X} := X|_{m,\pi_v \to p \to \dot{x}} = \left(Y^\mu - \frac{\dot{x}^\kappa \dot{x}_\kappa}{\dot{u}^2} f^\mu \right) \frac{\partial}{\partial x^\mu} \tag{17}$$

$$\tilde{I} := I|_{m,\pi_v \to p} = Y^\mu p_\mu - \frac{g^{\alpha\beta} f^\gamma p_\alpha p_\beta p_\gamma}{K^{\mu\nu} p_\mu p_\nu} \tag{18}$$

$\tilde{X}$ is higher order (hidden) Noether symmetry.

Conservation law: $\frac{d\tilde{I}}{d\tau} = \Sigma^\mu E_\mu(L) = 0$

Geometric interpretation of X

The distorted vector X satisfies (disformal transformation)

$$\mathcal{L}_X g_{\mu\nu} = 2\Omega(x)\left(g_{\mu\nu} + \frac{m^2}{\kappa}K_{\mu\nu}\right) \tag{19}$$

Theorem

If X satisfies Eq. (19), where $\nabla_{(\alpha}K_{\mu\nu)} = 0$, with $K^{\mu\nu}p_\mu p_\nu = \kappa$, then

$$I = X^\alpha p_\alpha \tag{20}$$

is conserved.

In the pp-wave case:

$$\kappa = \pi_v^2, \quad K = \partial_v \otimes \partial_v, \quad \Omega(x) = \omega_m$$

A different example

The de Sitter solution

$$ds^2 = -dt^2 + e^{\lambda t}\left(dx^2 + dy^2 + dz^2\right) \tag{21}$$

If we use

$$K_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{2\lambda t} & 0 & 0 \\ 0 & 0 & e^{2\lambda t} & 0 \\ 0 & 0 & 0 & e^{2\lambda t} \end{pmatrix} \tag{22}$$

Then,

$$\begin{aligned} X_0 &= \frac{e^{\frac{\lambda}{2}t}}{\left(\frac{m^2}{\kappa}e^{\lambda t}+1\right)^{\frac{1}{2}}}\partial_t, \quad X_i = \frac{e^{\frac{\lambda}{2}t}}{\left(\frac{m^2}{\kappa}e^{\lambda t}+1\right)^{\frac{1}{2}}}x^i\partial_t - \frac{2}{\lambda}e^{-\frac{\lambda}{2}t}\left(\frac{m^2}{\kappa}e^{\lambda t}+1\right)^{\frac{1}{2}}\partial_i \\ X_4 &= \frac{e^{-\frac{\lambda}{2}t}}{\left(\frac{m^2}{\kappa}e^{\lambda t}+1\right)^{\frac{1}{2}}}\left(\frac{4}{\lambda^2} - x^j x_j\right)\partial_t - \frac{4}{\lambda}e^{-\frac{\lambda}{2}t}\left(\frac{m^2}{\kappa}e^{\lambda t}+1\right)^{\frac{1}{2}}x^i\partial_i. \end{aligned} \tag{23}$$

The Bogoslovsky-Finsler case

The Finslerian generalization

Line element

$$ds^2 = F(x, dx)^2, \qquad (24)$$

where F is homogeneous of degree 1 in dx,

$$F(x, \lambda dx) = \lambda F(x, dx), \quad \lambda > 0.$$

ds^2 homogeneous of degree 2,
when ds^2 is quadratic in $dx \Rightarrow$ Riemannian geometry

A metric tensor can be introduced:

$$ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta, \quad G_{\alpha\beta}(x, dx) = -\frac{1}{2}\frac{\partial^2 F^2}{\partial(dx^\alpha)\partial(dx^\beta)} \qquad (25)$$

Bogoslovsky-Finsler generalization

For the pp-wave metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \left[\frac{K_{\mu\nu}dx^\mu dx^\nu}{-g_{\mu\nu}dx^\mu dx^\nu}\right]^b \tag{26}$$

where $K_{\mu\nu} = \ell_\mu \ell_\nu$, $\ell = \partial_v$.

[M. Elbistan, P.-M. Zhang, N.D., G. W. Gibbons, P. A. Horvathy, *Phys. Rev. D* **102**, 086016 (2020)]

In the Brinkmann coordinates

$$ds^2 = \left(-H(u,x,y)du^2 - 2dudv - \delta_{ij}dx^i dx^j\right)^{1-b} du^{2b} \tag{27}$$

$$\begin{cases} \text{In the case } H(u,x,y) = 0 \\ g_{\mu\nu} \text{ represents the flat space} \\ b \rightarrow \text{ Lorentz violation} \end{cases} \tag{28}$$

Finsler metric

$$G_{\mu\nu} = 2b(1-b)\left[g_{\sigma\mu}g_{\tau\nu}\frac{\mathcal{K}^b}{\mathcal{G}^{1+b}} + (g_{\sigma\mu}K_{\tau\nu} + g_{\sigma\nu}K_{\tau\mu})\frac{\mathcal{K}^{b-1}}{\mathcal{G}^b} + K_{\sigma\mu}K_{\tau\nu}\frac{\mathcal{K}^{b-2}}{\mathcal{G}^{1-b}}\right]\dot{x}^\sigma\dot{x}^\tau$$
$$(1-b)g_{\mu\nu}\frac{\mathcal{K}^b}{\mathcal{G}^b} - bK_{\mu\nu}\frac{\mathcal{K}^{b-1}}{\mathcal{G}^{b-1}}, \quad (29)$$

where $\mathcal{K} = K_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ and $\mathcal{G} = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$

Geodesic Lagrangian the einbein formalism:

$$L_b = \frac{1}{2N}G_{\mu\nu}(x,\dot{x})\dot{x}^\mu\dot{x}^\nu - N\frac{m^2}{2}$$
$$= -\frac{1}{2N}(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{(1-b)}(\ell_\mu\dot{x}^\mu)^{2b} - N\frac{m^2}{2} \quad (30)$$

Theorem

If K is a second rank covariantly constant Killing tensor of g, and there exists a vector X satisfying

$$\mathcal{L}_X\left(g_{\mu\nu} - \frac{b}{1-b}M_b^2 K_{\mu\nu}\right) = 2\Omega(x)\left(g_{\mu\nu} + M_b^2 K_{\mu\nu}\right), \tag{31}$$

where $M_b^{-2} = \frac{K_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$*, then the*

$$I = X^\mu p_\mu$$

is conserved along the geodesics.

[ND, arXiv preprint: 2205.14973 [gr-qc]]

The distorted vector

$$X = Y + \left[\frac{(1-b)^2 m^2}{\pi_v^2}\right]^{\frac{1}{1+b}} f_1 + \frac{b}{1-b}\left[\frac{(1-b)^2 m^2}{\pi_v^2}\right]^{\frac{1}{1+b}} f_2 \qquad (32)$$

where $f_1 = f$ under $\frac{m^2}{\pi_v^2} \to \left[\frac{(1-b)^2 m^2}{\pi_v^2}\right]^{\frac{1}{1+b}} =: M_b^2$,

$$\begin{aligned} f_2^u &= 0, \quad f_2^i = 0 \\ f_2^v &= \frac{\mu}{2}\delta_{ij}x^i x^j + a_i(u)x^i + a(u). \end{aligned} \qquad (33)$$

f_2 introduces a modification in the Killing sector.

The flat case example

The Bogoslovsky-Finsler line element.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \left[\frac{(\ell_\mu dx^\mu)^2}{-\eta_{\mu\nu} dx^\mu dx^\nu} \right]^b, \quad 0 < b < 1 \tag{34}$$

with $\nabla_\mu \ell^\nu = 0$, $\eta_{\mu\nu}\ell^\mu \ell^\nu = 0$, $\ell^0 > 0$.
In light-cone coordinates $(u, v, \mathrm{x}, \mathrm{y})$:

$$ds^2 = -(-dudv - d\mathrm{x}^2 - d\mathrm{y}^2)^{1-b}\,(du)^b$$

BF case	Minkowski $ds\|_{b=0}$
8d symmetry algebra: 7 out of the 10 KV of $\eta_{\mu\nu}$ + $\mathcal{N}_b = (b-1)u\partial_u + (1+b)v\partial_v + bx^i\partial_i$	10d symmetry algebra

Additional distorted symmetries

Non-distorted vectors (common symmetries):

$$T_u = \partial_u, \quad T_v = \partial_v, \quad T_i = \partial_i$$

$$R = x\partial_y - y\partial_x,$$

$$B_{vi} = u\partial_i - x_i\partial_v$$

From the Killing sector (KV of $\eta_{\mu\nu}$ when $b = 0$)

$$X_{uv} = B_{uv} + \frac{b}{1-b}M_b^2 u\partial_v = u\partial_u - v\partial_v + \frac{b}{1-b}M_b^2 u\partial_v$$

$$X_{ui} = B_{ui} + \frac{b}{1-b}M_b^2 x_i\partial_v = v\partial_i - x_i\partial_u + \frac{b}{1-b}M_b^2 x_i\partial_v.$$

Disformally related metric

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{b}{1-b}M_b^2 K_{\mu\nu} \tag{35}$$

Conserved quantities

The conserved quantities owed to X_{uv} and X_{ui}

$$I_{uv} = up_u - vp_v + \frac{bM_b^2 u}{1-b} p_v \tag{36a}$$

$$I_{ui} = vp_i - x_i p_u + \frac{bM_b^2 x_i}{1-b} p_v \tag{36b}$$

The actual Noether charges of the corresponding hidden symmetries

$$\tilde{I}_{uv} = vp_v - up_u + \frac{b}{1+b} u \frac{p_\mu p^\mu}{p_v} \tag{37a}$$

$$\tilde{I}_{ui} = vp_i - x_i p_u + \frac{b}{1+b} x_i \frac{p_\mu p^\mu}{p_v}. \tag{37b}$$

The proper CKV sector (proper CKVs of $\eta_{\mu\nu}$ when $M_b = 0$)

$$
\begin{aligned}
X_h &= \left(M_b^2 u + 2v\right)\partial_v + x^i\partial_i \\
X_{C_u} &= u^2\partial_u + \frac{1}{2}\left(\frac{1+b}{1-b}M_b^2 u^2 - \mathbf{x}^2\right)\partial_v + ux^i\partial_i \\
X_{C_v} &= \frac{\mathbf{x}^2}{2}\partial_u - \frac{1}{4}\left[M_b^4 u^2 + M_b^2\left(4uv - \frac{1-b}{1+b}\mathbf{x}^2\right) + 4v^2\right]\partial_v - \left(\frac{M_b^2}{2}u + v\right)x^i\partial_i \\
X_{C_j} &= ux_j\partial_u + \left(\frac{1}{1-b}M_b^2 u + v\right)x_j\partial_v - \frac{1}{2}\left(M_b^2 u^2 + 2uv + \mathbf{x}^2\right)\partial_j + x_j x^i\partial_i .
\end{aligned}
$$

where $\mathbf{x} = \mathrm{x}^2 + \mathrm{y}^2$.

For all of the above we have conserved charges

$$I = X^\mu p_\mu \tag{38}$$

which is a reduced form of the hidden symmetry charges

Conclusion

Concluding remarks

- Explicit symmetry breaking owed to m, b
- Existence of a larger class of symmetries depending continuously on m, b
- In pp-waves in the Lorentzian case
  - Higher order symmetries involving CKVs
  - Can be seen as a mass dependent distortion of the CKVs
- In the Bogoslovsky-Finsler generalization
  - Additional b-distortion is introduced
  - The modification now can affect Killing vectors