Symmetry operators for the conformal wave equation in rotating black hole spacetimes

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Outline

1. Hidden symmetries, integrability, and separability
2. Conformal symmetry operators
3. Rotating black holes
4. Conformal wave equation in black hole spacetimes

Hidden symmetries, integrability, and separability

Symmetries in phase space

Symplictic manifold $(\mathcal{M}, \omega)$ w/ Darboux coords (x^μ, p_μ)

- Symplectic form $\omega = dp_\mu \wedge dx^\mu$, $\omega^{ab} \equiv (\omega^{-1})^{ab}$
- Hamiltonian vector & Poisson brackets

$$i_{X_f}\omega = -df \iff X_f = \omega^{ab}\partial_b f$$

$$\{f, g\} \equiv \omega(X_f, X_g)$$

- Noether's theorm: Conserved quantity Q of Hamiltonian H

$$\{Q, H\} = 0$$

$\implies$ Symmetry $([\delta_Q x]^\mu, [\delta_Q p]_\mu)$ generated by Hamiltonian vector

$$X_Q = [\delta_Q x]^\mu \frac{\partial}{\partial x^\mu} + [\delta_Q p]_\mu \frac{\partial}{\partial p_\mu}$$

Symmetries: Hidden and Explicit

Particle motion $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ and geodesics $p^\mu \nabla_\mu p_\nu = 0$

- Linear charge $\iff$ Killing equation: Projectable Hamiltonian vector (Explicit)

$$Q_\xi = \xi^\mu p_\mu \iff \nabla_{(\mu}\xi_{\nu)} = 0\,, \quad \pi_*(X_{Q_\xi}) = \xi^\mu \frac{\partial}{\partial x^\mu}$$

- Higher order $\iff$ Killing tensor equation

$$Q_K = K^{\mu_1 \dots \mu_s} p_{\mu_1} \dots p_{\mu_s} \iff \nabla_{(\mu} K_{\mu_1 \dots \mu_s)} = 0$$

- *Not* projectable $\iff$ Dynamical/Hidden

$$\pi_*(X_{Q_k}) = s\, K^{\mu\mu_1 \dots \mu_{s-1}} p_{\mu_1} \dots p_{\mu_{s-1}} \frac{\partial}{\partial x^\mu}$$

- Killing–Yano forms $f_{a_1 \dots a_p}$: $\nabla_a f_{a_1 \dots a_p} = \nabla_{[a} f_{a_1 \dots a_p]}$
  $\implies$ Square to Killing tensors

$$K^f_{ab} \propto f_{ac_1 \dots c_{s-1}} f_b{}^{c_1 \dots c_{s-1}}$$

Conformal Symmetries

Can generalize these symmetries to the conformal class of spacetimes $g \to \Omega^2 g$:

- Conformal Killing vectors

$$\nabla_{(\mu}\xi_{\nu)} = \alpha g_{\mu\nu}$$

- Conformal Killing tensors

$$\nabla_{(\mu} K_{\mu_1 \dots \mu_s)} = g_{(\mu\mu_1}\alpha_{\mu_2 \dots \mu_s)}$$

- Conformal Killing-Yano tensor

$$\nabla_a f_{a_1 \dots a_p} = \nabla_{[a} f_{a_1 \dots a_p]} + \frac{p}{D-p+1} g_{a[a_1} \nabla^b f_{|b|a_2 \dots a_{p-1}]}$$

Important—Principal tensor h:
non degenerate closed conformal Killing–Yano (CCKY) 2 form

$$\nabla_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b$$

Integrability and Separability

- Geodesic motion, $p^{\mu}\nabla_{\mu}p_{\nu} = 0$, integrable $\iff$ D constants of motion, e.g. maximally symmetric spacetimes
- Related: Field equations solved by separation of variables $\iff$ D mutually commuting symmetry operators[1]
- Rotating black holes: 2 Killing vectors *and* one non-trivial Killing tensor

[1] Sym. Op. $\iff$ $\mathcal{Q} \circ \text{E.o.M} = \text{E.o.M} \circ \mathcal{Q}$

Conformal symmetry operators

Conformal wave equation

- Scalar field Φ with E.o.M

$$\Box_R \, \Phi \equiv \left(\Box - \frac{D-2}{4(D-1)} R \right) \Phi = 0$$

- Conformal (Weyl) scaling $g_{ab} \to \widetilde{g}_{ab} \equiv \Omega^2 g_{ab}$

$$\Box_R \to \widetilde{\Box}_R = \Omega^{-2+w} \circ \Box_R \circ \Omega^{-w} \,, \quad w \equiv 1 - D/2$$

- E.o.M. invariant if we also scale $\quad \Phi \to \widetilde{\Phi} = \Omega^w \, \Phi$

$$\widetilde{\Box}_R \, \widetilde{\Phi} \equiv \Omega^{w-2} \, \Box_R \, \Phi = 0$$

What are its symmetry operators $\mathcal{Q}$?

$$\mathcal{Q} \circ \Box_R = \Box_R \circ \mathcal{Q}$$

First construction of symmetry operators

- Killing vectors provide natural examples:

$$\mathcal{Q}(\xi) = -i\xi^a \nabla_a$$

- What about Killing tensors?

$$\mathcal{Q}(K) = \nabla^a K^{ab} \nabla_b + f$$

- The commutator $[\mathcal{Q}_K, \Box_R]$ vanishes iff[2][3] geometric condition:

$$K^{ab}\nabla_b R + \frac{1}{3}\nabla_b \left(K^{bc} R^a_c - R^{bc} K^a_c\right) = \nabla^a f$$

Can this be generalized to the entire conformal class of spacetimes?

[2] S. Benenti et al. J. Math. Phys. 43, 5223 (2002).
[3] B. Carter, Phys. Rev. D 16, 3395, (1977).

Detour: Conformal geometry

Promote the scalar functions to densities[4] of weight λ;
$\phi \to |g|^{\lambda/2}\phi \in \mathcal{F}_\lambda$

- Consider conformally invariant differential operators

$$\mathcal{D}_{\lambda,\mu} : \mathcal{F}_\lambda \to \mathcal{F}_\mu$$

- e.g. Conformal Laplacian $\Box_R : \mathcal{F}_{\lambda_0} \to \mathcal{F}_{\mu_0}$ maps weights

$$\lambda_0 = -w/D \to \mu_0 = (2-w)/D$$

$$\begin{array}{ccc}
\mathcal{F}_\lambda(M) & \xrightarrow{\mathcal{D}_{\lambda,\mu}} & \mathcal{F}_\mu(M) \\
{\scriptstyle |g|^{\lambda/2}} \uparrow & & \uparrow {\scriptstyle |g|^{\mu/2}} \\
\mathcal{C}^\infty(M) & \xrightarrow[|g|^{-\mu/2}\circ\mathcal{D}_{\lambda,\mu}\circ|g|^{\lambda/2}]{} & \mathcal{C}^\infty(M)
\end{array}$$

- Symmetry operators are now: $\mathcal{Q}_{\mu_0,\mu_0} \circ \Box_R = \Box_R \circ \mathcal{Q}_{\lambda_0,\lambda_0}$

[4](Michel, Radoux, and Šilhan, SIGMA, (2014), arxiv:1308.1046)

Conformal Sym. Ops. as "quantisation map": second order

- Given any symmetric S^{ab} tensor density of weight $\delta = \mu - \lambda$
  $\rightarrow$ most general conformally invariant operator:[5]

$$\begin{aligned}\mathcal{Q}_{\lambda,\mu}(S) = & \nabla_a S^{ab} \nabla_b + \Big(\gamma_1[\nabla_a S^{ab}] + \gamma_2[\nabla^b \operatorname{Tr} S]\Big)\nabla_b \\ & + \gamma_3(\nabla_a \nabla_b S^{ab}) + \gamma_4(\Box \operatorname{Tr} S) + \gamma_5\, R_{ab} S^{ab} \\ & + \gamma_6\, R \operatorname{Tr} S + f\end{aligned}$$

  where γ_i are consts. depending on λ, μ, and D
- As a map on functions[6] $\mathcal{Q}_{\lambda\,,\mu}(S)$ transforms as:

$$\widetilde{\mathcal{Q}}_{\lambda,\mu} \equiv \Omega^{-\mu D} \circ Q_{\lambda,\mu} \circ \Omega^{\lambda D}$$

  when $g_{ab} \rightarrow \widetilde{g}_{ab} = \Omega^2 g_{ab}$

[5](Michel, Radoux, and Šilhan, SIGMA, (2014), arxiv:1308.1046)
[6]Here I abuse notation considering the isomorphism between operators on scalars and on scalar densities

Theorem: Conformal symmetry operators of $\Box_R$

Theorem 1.[7] *Let K^{ab} be a (special) Killing tensor of metric $\boldsymbol{g}$, so that the following conformally invariant 'geometric obstruction' built from the Weyl tensor C_{abcd}:*

$$Obs(K)_a = \frac{2(D-2)}{3(D+1)}\left(\nabla_b K^{cd} C^b{}_{cda} - \frac{3}{D-3} K^{cd} \nabla_b C^b{}_{cda}\right)$$

is exact,

$$\mathbf{Obs}(K) = -2\boldsymbol{d}f\,.$$

Then $\mathcal{Q}_{\lambda,\lambda}(K)$ (with f defined by above) is a symmetry operator for the conformal wave operator and in fact satisfies

$$\mathcal{Q}_{\mu_0,\mu_0} \circ \Box_R = \Box_R \circ \mathcal{Q}_{\lambda_0,\lambda_0}$$

[7] (Michel, Radoux, and Šilhan, SIGMA, (2014), arxiv:1308.1046)

Rotating black holes

Kerr black hole

1963, Roy P. Kerr: Boyer–Linquist form

$$\boldsymbol{g} = -\left(1 - \frac{2Mr}{\Sigma}\right) \boldsymbol{d}t^2 - \frac{4Mra\sin^2\theta}{\Sigma}\, \boldsymbol{d}t\, \boldsymbol{d}\phi + \frac{A \sin^2\theta}{\Sigma}\, \boldsymbol{d}\phi^2 + \frac{\Sigma}{\Delta_r}\, \boldsymbol{d}r^2 + \Sigma\, \boldsymbol{d}\theta^2$$

$$\Sigma = r^2 + a^2 \cos^2\theta\,, \quad \Delta_r = r^2 - 2Mr + a^2\,, \quad A = (r^2+a^2)^2 - \Delta_r\, a^2 \sin^2\theta$$

- Killing vectors ∂_t, ∂_ϕ
- Killing tensor K^{ab} leads to Carter's constant
- Geodesics are integrable
- Scalar, spinor, vector, and tensor field equations all separate (going back to 70s[8])

[8] See (Frolov, Krtouš, and Kubizňák, Liv. Living Rev. Relativ. (2017), arXiv:1705.5482) for modern/historical refs.

Kerr–NUT–AdS in all dimensions $D = 2n + \varepsilon$...

- Metric

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{n-1} A_\mu^{(j)} \boldsymbol{d}\psi_j \Big)^2 \right] + \frac{\varepsilon c}{A^{(n)}} \Big(\sum_{k=0}^{n} A^{(k)} \boldsymbol{d}\psi_k \Big)^2 ,$$

  $A^{(k)}$, $A_\mu^{(j)}$, and U_μ are 'symmetric polynomials' of x_μ
- Coordinates ψ_k are Killing directions

$$\boldsymbol{l}_{(k)} = \boldsymbol{\partial}_{\psi_k}$$

  and x_μ are angular/radial[9]
- Metric funcs. off shell/arbitrary

$$X_\mu = X_\mu(x_\mu)$$

- Ricci scalar is simple

$$R = \sum_{\mu=1}^{n} \frac{r_\mu}{U_\mu} , \quad r_\mu = -X_\mu'' - \frac{2\varepsilon X_\mu'}{x_\mu} - \frac{2\varepsilon c}{x_\mu^4}$$

[9] $\psi_0 \sim \tau$, $x_n \sim ir$, $x_\mu \sim \theta_i$

Killing tower

Principal Killing vector $l_{(0)} \equiv \xi$

- *principal* tensor $\boldsymbol{h}$: non-degenerate closed conformal Killing–Yano (CCKY) 2-form

$$\nabla_a h_{bc} = g_{ab}\,\xi_c - g_{ac}\,\xi_b\,, \quad \xi_a = \frac{1}{D-1}\nabla^b h_{ba}$$

- '*symmetry descendants*' of $\boldsymbol{h}$: Tower of CCKY tensors:

$$\boldsymbol{h}^{(j)} = \frac{1}{j!}\underbrace{\boldsymbol{h}\wedge\cdots\wedge\boldsymbol{h}}_{j\ \text{times}}\,.$$

- Hodge duals $\boldsymbol{f}^{(j)} = \star\boldsymbol{h}^{(j)}$ are Killing–Yano tensors: gives tower of Killing tensors:

$$k^{ab}_{(j)} \propto f^{(j)a}{}_{c_1\ldots d_{d-2j-1}} f^{(j)bc_1\ldots c_{d-2j-1}}$$

- tower of Killing vectors:

$$l^a_{(j)} = k^{ab}_{(j)}\,\xi_b$$

Kerr–NUT–AdS: Separability and integrability properties

- D symmetries: $n + \epsilon$ Killing vectors and n Killing tensors
- Geodesic motion integrable
- Scalar, spinor, vector fields all separate!
- Role of h important but not always explict.

What about the conformal wave equation?

Conformal wave equation in black hole spacetimes

Conformal wave equation in Kerr–NUT–AdS

Explicit coordinate calculations:

- Recall the wave equation

$$\Box_R \Phi \equiv \left(\nabla_a g^{ab} \nabla_b - \frac{D-2}{4(D-1)} R \right) \Phi = 0$$

  and Ricci scalar

$$R = \sum_{\mu=1}^{n} \frac{r_\mu}{U_\mu}$$

- Suggestive: $k^{ab}_{(0)} = g^{ab}$ and $A^{(0)}_\mu = 1$ define

$$\mathcal{K}_{(0)} \equiv \Box_R = \nabla_a k^{ab}_{(0)} \nabla_b - \frac{D-2}{4(D-1)} \sum_{\mu=1}^{n} \frac{A^{(0)}_\mu r_\mu}{U_\mu}$$

Symmetry operators for Kerr–NUT–AdS

- complete set of symmetry operators $j \in \{0, n-1\}$[10]

$$\mathcal{K}_{(j)} = \nabla_a k^{ab}_{(j)} \nabla_b - \frac{D-2}{4(D-1)} R_{(j)}\,, \quad R_{(j)} = \sum_{\mu=1}^{n} \frac{A^{(j)}_\mu}{U_\mu} r_\mu$$

$$\mathcal{L}_{(j)} = -\, i\, l^a_{(j)} \nabla_a$$

- The extra functions satisfy $\nabla_a R_{(j)} = k^{(j)}_{ab} \nabla^b R$
- Mutually commuting since all generated from a single object h:

$$\left[\mathcal{K}_{(k)}, \mathcal{L}_{(l)}\right] = 0\,, \quad \left[\mathcal{L}_{(k)}, \mathcal{L}_{(l)}\right] = 0\,, \quad \left[\mathcal{K}_{(k)}, \mathcal{K}_{(l)}\right] = 0$$

- $\implies$ common eigenfunction basis $\Phi_{C,L}$

$$\mathcal{K}_{(j)} \Phi_{C,L} = C_j\, \Phi_{C,L}\,, \quad \mathcal{L}_{(j)} \Phi_{C,L} = L_j\, \Phi_{C,L}$$

[10] F.G. et al, PRD 2020, arXiv:2002.05221

Separability

Explicitly seen with:

- Ansatz

$$\Phi = \prod_{\mu=1}^{n} Z_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i\, L_k \psi_k}$$

- E.o.M becomes uncoupled ODEs

$$Z''_\mu + Z'_\mu \left(\frac{X'_\mu}{X_\mu} + \frac{\varepsilon}{x_\mu} \right) - \frac{Z_\mu}{X_\mu^2} \left(\sum_{k=0}^{n-1+\varepsilon} (-x_\mu^2)^{n-1-k} L_k \right)^2$$
$$- \frac{Z_\mu}{X_\mu} \left(\frac{D-2}{4(D-1)} r_\mu(x_\mu) + \frac{\varepsilon}{c x_\mu^2} L_n^2 + \sum_{k=0}^{n-1} C_k (-x_\mu^2)^{n-1-k} \right) = 0$$

- In the conformal spacetime $\implies$ R-separated solution $\widetilde{\Phi} = \Omega^w \, \Phi$
  Can we intrinsically characterise its separability?

Conformal symmetry operators in Kerr–NUT–AdS

- By **Theorem 1.**[11] coordinate form $\mathcal{Q}_{\lambda_0,\lambda_0}(k_{(j)})$ and $\mathcal{K}_{(j)}$ must be equivalent (for $\Omega = 1$ frame and $j > 0$)

- Can check for each $j > 0$ the obstruction is exact: $\mathbf{Obs}(k_{(j)}) = -2\boldsymbol{d} f_{(j)}$ for some $f_{(j)}$
- By construction mutually commute, e.g.

$$\left[\widetilde{\mathcal{Q}}(k^{(i)})_{\lambda_0,\lambda_0}, \widetilde{\mathcal{Q}}(k^{(j)})_{\lambda_0,\lambda_0}\right] = 0$$

- This is what geometrically underlies R-separability in full conformally related class to Kerr–NUT–AdS!

$$\widetilde{\Phi} = \Omega^w \, \Phi = \Omega^w \prod_{\mu=1}^{n} Z_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i\, L_k \psi_k}$$

[11](Michel, Radoux, and Šilhan, SIGMA, (2014), arxiv:1308.1046)

Summary

- Hidden symmetries lead to integrability and separability
- Conformal symmetry operators for wave equation
- Kerr–NUT–AdS is *very* special
- Symmetry operators for conformal wave equation in any frame

Extensions:

- Conformally invariant "quantization map" for higher spins?
- Find the role of principal tensor in separation for gravitons–Hard!

Thank you!—Questions?

Extras

Killing objects

- Killing vector

$$\nabla_{(a}k_{b)} = 0$$

- Killing tensor

$$\nabla_{(a}K_{a_1\dots a_s)} = 0$$

- Killing–Yano tensor

$$f_{(a_0 a_1\dots a_s)} = 0\,,\quad \nabla_{(b}f_{a_0)a_1\dots a_s} = 0$$

- Conformal Killing vector

$$\nabla_{(a}q_{b)} = \alpha g_{ab}$$

- Conformal Killing tensor

$$\nabla_{(a}Q_{a_1\dots a_s)} = g_{(a_0 a_1}\alpha_{a_2\dots a_s)}$$

- Closed conformal Killing–Yano form

$$\boldsymbol{d}h = 0\,,\quad \nabla_b h_{a_1\dots a_s} = \frac{s}{D-s+1} g_{a[a_1}\nabla^b h_{|b|a_2\dots a_s]}$$

Kerr black hole 2

Recall Boyer–Lindquist coords:

$$\boldsymbol{g} = -\left(1-\frac{2Mr}{\Sigma}\right)\boldsymbol{d}t^2 - \frac{4Mra\sin^2\theta}{\Sigma}\,\boldsymbol{d}t\,\boldsymbol{d}\phi + \frac{A\,\sin^2\theta}{\Sigma}\,\boldsymbol{d}\phi^2 + \frac{\Sigma}{\Delta_r}\,\boldsymbol{d}r^2 + \Sigma\,\boldsymbol{d}\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta\,,\quad \Delta_r = r^2 - 2Mr + a^2\,,\quad A = (r^2+a^2)^2 - \Delta_r\,a^2\sin^2\theta$$

- Coordinate transform $y = a\cos\theta\,, \psi = \phi/a\,, \tau = t - a\phi$ and set $\Delta_y = a^2 - y^2$: "Canonical"/Carter's form

$$\boldsymbol{g} = \frac{1}{\Sigma}\left[-\Delta_r(\boldsymbol{d}\tau + y^2\boldsymbol{d}\psi)^2 + \Delta_y(\boldsymbol{d}\tau - r^2\boldsymbol{d}\psi)^2\right] + \Sigma\left[\frac{\boldsymbol{d}r^2}{\Delta_r} + \frac{\boldsymbol{d}y^2}{\Delta_y}\right]$$

- This generalises to higher D "Wick" rotating $r \to ix$ for symmetric form

Kerr-NUT–AdS full gory details

Metric

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu}\, \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{n-1} A_\mu^{(j)}\, \boldsymbol{d}\psi_j \Big)^2 \right] + \frac{\varepsilon c}{A^{(n)}} \Big(\sum_{k=0}^{n} A^{(k)}\, \boldsymbol{d}\psi_k \Big)^2$$

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1 \\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \dots x_{\nu_k}^2 \,, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\dots<\nu_j \\ \nu_i \neq \mu}}^{n} x_{\nu_1}^2 \dots x_{\nu_j}^2 \,,$$

$$U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2) \,, \quad U = \prod_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{n} (x_\mu^2 - x_\nu^2) = \det\big(A_\mu^{(j)}\big) \,, \tag{1}$$

On shell of $G_{ab} + \Lambda g_{ab} = 0$,

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2 b_\mu x_\mu^{1-\varepsilon} - \frac{\varepsilon c}{x_\mu^2} \,, \tag{2}$$

$\Lambda = \frac{1}{2}(-1)^n (d-1)(d-2) c_n$ and c_k, b_μ, c are related to rotations, mass, and NUT charges