# Triangular decoupling of harmonic gauge modes in linearized gravity on a Schwarzschild black hole

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**2004.09651** $\rightsquigarrow$ SIGMA **18** 011 (2022)

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**A2:** Mathematical Relativity
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# Harmonic Gauge and its Advantages

- On a Lorentzian $(M, g)$, $R_{\mu\nu} = 0$ vacuum, consider scalar $z$ ($s = 0$), Maxwell $v_\mu$ ($s = 1$) and Einstein $p_{\mu\nu}$ ($s = 2$) perturbations:

$$(SW) \quad \Box z = 0,$$

$$(Max)\ (VW) \quad \Box v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0$$

$$(v_\mu = \nabla_\mu z \rightsquigarrow \Box z = 0 \text{ residual gauge dynamics}),$$

$$(Ein)\ (LW) \quad \Box p_{\mu\nu} - 2\, R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\, \nabla_{(\mu} \nabla^\lambda \overline{p}_{\nu)\lambda} = 0$$

$$(p_{\mu\nu} = \nabla_{(\mu} v_{\nu)} \rightsquigarrow \Box v_\mu = 0 \text{ residual gauge dynamics}).$$

- Under harmonic gauges ($\nabla^\mu v_\mu = 0$ and $\nabla^\nu \overline{p}_{\mu\nu} = \nabla^\nu (p_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \operatorname{tr} p) = 0$) we get the vector wave and Lichnerowicz wave equations.
- **Advantages** over more popular gauges.
  (Regge-Wheeler, radiation, double null, ...)
  - Geometrically natural (obeys nice identities).
  - Nice local analytic behavior (Green function has Hadamard form).
  - Applications in QFT and gravitational wave modelling.

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  - Yes, mode stable in Regge-Wheeler gauge. (Regge, Wheeler, Zerilli, . . . 1950s+)
  - Yes, with global boundedness in Regge-Wheeler (gauge invariant) variables (Dotti 2014).
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# Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit $\omega lm$):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi) e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-$s$ Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

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- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C} \setminus \mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
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# Spectral Problem in Triangular Form

## Lemma (△)

*$\tilde{E}_\omega$ has a positive self-adjoint $\omega^2$-spectral problem on $L^2(2M,\infty;\frac{dr}{f})^{\oplus 7}$,*

$$\tilde{E}_\omega = \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix},$$

*provided $\|\mathcal{D}_s{}^{-1}(*)\| < \infty$ (is relatively bounded). ($\tilde{E}_\omega \rightsquigarrow$ any triang.op.)*

**Proof:** $\tilde{E}_\omega^{-1}$ is polynomial in $\mathcal{D}_s{}^{-1}$ and $\mathcal{D}_s{}^{-1}(*)$. □

**Strategy:** Try to use (△) on the radial mode equations of

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*provided $\|\mathcal{D}_s{}^{-1}(*)\| < \infty$ (is relatively bounded). ($\tilde{E}_\omega \rightsquigarrow$ any triang.op.)*

**Proof:** $\tilde{E}_\omega^{-1}$ is polynomial in $\mathcal{D}_s{}^{-1}$ and $\mathcal{D}_s{}^{-1}(*)$. $\square$

**Strategy:** Try to use ($\triangle$) on the radial mode equations of

$$(\mathit{Max})\ (\mathit{VW}) \quad \Box_{\mathbf{g}} v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0,$$

$$(\mathit{Ein})\ (\mathit{LW}) \quad \Box_{\mathbf{g}} p_{\mu\nu} - 2\, R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\, \nabla_{(\mu} \nabla^\lambda \overline{p}_{\nu)\lambda} = 0,$$

harmonic gauge Maxwell $v_\mu$ ($s = 1$) and Einstein $p_{\mu\nu}$ ($s = 2$) pert.s.

# Obstacle:

# Radial Mode Equation: $VW_\omega[v] = 0$

Explicitly, $v_\mu \to v(r) = (v_t, v_r, u \mid w)$:

(odd) $$\partial_r \mathcal{B}_l r^2 f \partial_r w + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \mathcal{B}_l w + \mathcal{B}_l \frac{2M}{r} w = 0,$$

(even) $$\begin{bmatrix} -\partial_r \frac{1}{f} r^2 f \partial_r v_t \\ \partial_r f r^2 f \partial_r v_r \\ \partial_r \mathcal{B}_l r^2 f \partial_r u \end{bmatrix} + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \begin{bmatrix} -\frac{1}{f} v_t \\ f v_r \\ \mathcal{B}_l u \end{bmatrix}$$
$$+ i\omega \frac{2M}{f} \begin{bmatrix} v_r \\ -v_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2f^2 & 2\mathcal{B}_l f \\ 0 & 2\mathcal{B}_l f & \mathcal{B}_l \frac{2M}{r} \end{bmatrix} \begin{bmatrix} v_t \\ v_r \\ u \end{bmatrix} = 0,$$

where $f(r) = 1 - \frac{2M}{r}$ and $\mathcal{B}_l = l(l+1)$.

# Radial Mode Equation: $LW_\omega[p] = 0$ (odd sector)

Explicitly, $p_{\mu\nu} \to p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2)$:

$$\begin{bmatrix} \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f\partial_r)\, h_t \\ \partial_r(2\mathcal{B}_l f\, r^2 f\partial_r)\, h_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f\partial_r)\, h_2 \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -4\frac{\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 \\ 0 & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix}$$

$$-i\omega\frac{4M}{f} \begin{bmatrix} 0 & -\mathcal{B}_l & 0 \\ \mathcal{B}_l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix} + \omega^2 \frac{r^2}{f} \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix} = 0$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

# Radial Mode Equation: $LW_\omega[p] = 0$ (even sector)

$$
\begin{bmatrix}
\partial_r(-2\,r^2 f\partial_r)\,h_{tr} \\
\partial_r(-2\frac{\mathcal{B}_l}{f}\,r^2 f\partial_r)\,j_t \\
\partial_r(\frac{1}{f^2}\,r^2 f\partial_r)\,h_{tt} \\
\partial_r(f^2\,r^2 f\partial_r)\,h_{rr} \\
\partial_r(2\,r^2 f\partial_r)\,K \\
\partial_r(2\mathcal{B}_l f\,r^2 f\partial_r)\,j_r \\
\partial_r(\frac{\mathcal{A}_l}{2}\,r^2 f\partial_r)\,G
\end{bmatrix}
- \mathcal{B}_l
\begin{bmatrix}
-2\,h_{tr} \\
-2\frac{\mathcal{B}_l}{f}\,j_t \\
\frac{1}{f^2}\,h_{tt} \\
f^2\,h_{rr} \\
2\,K \\
2\mathcal{B}_l f\,j_r \\
\frac{\mathcal{A}_l}{2}\,G
\end{bmatrix}
$$

$$
\begin{bmatrix}
\frac{2(f^2+1)}{f} & -4\mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\
-4\mathcal{B}_l & -\frac{4\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4M^2}{2f^3 r^2} & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{r} & \frac{2}{f}\frac{2M}{r} & 0 & 0 \\
0 & 0 & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{e} & \frac{f(\frac{4M^2}{r^2}-8f^2)}{2} & 4f(1-\frac{3M}{r}) & 4\mathcal{B}_l f^2 & 0 \\
0 & 0 & \frac{2}{f}\frac{2M}{r} & 4f(1-\frac{3M}{r}) & -4(1-\frac{4M}{r}) & -4\mathcal{B}_l f & 0 \\
0 & 0 & 0 & 4\mathcal{B}_l f^2 & -4\mathcal{B}_l f & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\
0 & 0 & 0 & 0 & 0 & 2\mathcal{A}_l f & \mathcal{A}_l
\end{bmatrix}
\begin{bmatrix}
h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G
\end{bmatrix}
$$

$$
-i\omega\frac{4M}{f}
\begin{bmatrix}
0 & 0 & -\frac{1}{f} & -f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\mathcal{B}_l & 0 \\
\frac{1}{f} & 0 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G
\end{bmatrix}
+ \omega^2\frac{r^2}{f}
\begin{bmatrix}
-2\,h_{tr} \\
-2\frac{\mathcal{B}_l}{f}\,j_t \\
\frac{1}{f^2}\,h_{tt} \\
f^2\,h_{rr} \\
2\,K \\
2\mathcal{B}_l f\,j_r \\
\frac{\mathcal{A}_l}{2}\,G
\end{bmatrix}
= 0
$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

# Equivalence to Triangular Form

(**N.B.:** All equations and operators have $r$-rational coefficients.)

## Definition (Equivalence)

Differential equations are equivalent, $E_\omega[u] = h \sim \tilde{E}_\omega[\tilde{u}] = \tilde{h}$, when differential operators map solutions to solutions $(u, h) \leftrightsquigarrow (\tilde{u}, \tilde{h})$, while bijective on-shell $(h = \tilde{h} = 0)$.

(**Context:** see 2004.09651 or `youtu.be/dy-QO5NFHC0` for details.)

## Lemma (Spectral Equivalence for radial mode eqs.)

*If $E_\omega \sim \tilde{E}_\omega$ for $\omega \notin \mathbb{R} \subset \mathbb{C}$, then when one has a real $\omega$-spectral problem so does the other.*

## Proof.

The spectrum of a radial mode equation $E_\omega[u] = 0$ or $\tilde{E}_\omega[\tilde{u}] = 0$ excludes $\omega \in \mathbb{C}$ when there are no bound states with $u \sim rat(r)\, e^{i\omega r_*}$ asymptotics. The equivalence differential operators are bijective on solutions and preserve such asymptotics. □

# Final Result:

# Final Reduced Decoupled Forms

- Vector wave equation ($l > 0$) 1711.00585:
  - $VW^{\text{odd}}_{\omega} \sim \mathcal{D}_1$ $\quad VW^{\text{even}}_{\omega} \sim$
- Lichnerowicz wave equation ($l > 1$) 2004.09651:
  - $LW^{\text{odd}}_{\omega} \sim$
  - $LW^{\text{even}}_{\omega} \sim$
- **NEW:** completes previous partial and ad hoc results. [Berndtson (PhD, 2007)]

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- Vector wave equation ($l > 0$) 1711.00585:
  - $VW^{\text{odd}}_{\omega} \sim \mathcal{D}_1 \quad VW^{\text{even}}_{\omega} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{2r}\right) \\ 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- Lichnerowicz wave equation ($l > 1$) 2004.09651:
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- Lichnerowicz wave equation ($l > 1$) 2004.09651:
  - $LW^{\mathrm{odd}}_\omega \sim \begin{bmatrix} \mathcal{D}_1 & 0 & \frac{2M}{r^3}\frac{\mathcal{B}_l}{3} \\ 0 & \mathcal{D}_2 & 0 \\ 0 & 0 & \mathcal{D}_1 \end{bmatrix}$
  - $LW^{\mathrm{even}}_\omega \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{M^2}{2r^4}(7\mathcal{B}_l + 2) \\ 0 & \mathcal{D}_1 & 0 & 0 & 0 & -\frac{2M}{r^3}\frac{5\mathcal{B}_l}{3} & 0 \\ 0 & 0 & \mathcal{D}_0 & 0 & 0 & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & \mathcal{D}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & 0 & 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
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# Equivalence Strategy:

# Strategy: triangular decoupling and reduction

$$E_\omega = \begin{bmatrix} & & \\ & LW_\omega & \\ & & \end{bmatrix}$$

- mode hierarchy
  $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, $VW$ and $LW$ are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$ (into pure gauge, gauge invariant and constraint violating modes.)
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

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$$E_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ & \mathcal{D}_2 & * \\ & & VW_\omega \end{bmatrix}$$

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# Discussion

- A systematic decoupling and reduction strategy reduces a complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- The results have applications to spectral problems, hidden symmetries and Debye potentials.
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr black hole?

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Thank you for your attention!