The Cauchy problem of pp-waves

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Instituto de Física Teórica UAM-CSIC

Introduction

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Yes!

[Choquet-Bruhat '52; Choquet-Bruhat, Geroch '69].

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The GR initial value problem is well-defined on **globally hyperbolic manifolds**[1]. These are spacetimes endowed with a spacelike hypersurface which every non-spacelike curve intersects exactly once, called **Cauchy surface**.

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By a celebrated result of Bernal and Sánchez, **[Bernal, Sánchez '04]**, the **isometry type** of a globally hyperbolic four-manifold (M, g) with Cauchy surface Σ is given by:

$$(M, g) = \left(\mathcal{I} \times \Sigma, -\beta_t^2 \mathrm{d}t^2 + h_t\right),$$

where $\mathcal{I} \subset \mathbb{R}$, $t \in \mathcal{I}$ is identified with time, $\{h_t\}_{t\in\mathcal{I}}$ is a family of Riemannian metrics on Σ and $\{\beta_t\}_{t\in\mathbb{R}}$ is a family of positive functions on Σ.

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At the initial time $t = 0$, define $h = h_t|_{t=0}$ and:

$$\Theta = -\frac{1}{2\beta_t}\partial_t h_t\Big|_{t_0} \,.$$

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If (Σ, h, Θ) is a triple satisfying the GR constraint equations $\to$ unique Ricci flat maximal Cauchy development (M, g) such that the induced metric and the shape operator on Σ are h and Θ, respectively **[Choquet-Bruhat '52; Choquet-Bruhat, Geroch '69]**.

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2. Parallel spinors appear naturally within the study of **supersymmetric backgrounds** in Supergravity and String Theory.
3. These spacetimes correspond to **pp-wave solutions**, characterized by admitting a parallel one-form u and $\mathrm{Ric}^g = f\, u \otimes u$. Consequently, studying the **evolution problem** of a **real parallel spinor** is **equivalent** to exploring the **Cauchy problem of pp-waves**.

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3. We show that the **Ricci-flat flow** and the **parallel spinor flow coincide on common initial data**, which in turn provides an **initial data characterization** of Ricci flat **pp-waves**.
4. We define the notion of **left-invariant parallel spinor flows** on simply connected Lie groups, showing an explicit example.

Real parallel spinors

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To this aim, it is extremely convenient to use the formalism of **parabolic pairs**, according to which a real parallel spinor on four Lorentzian dimensions is equivalent to an algebraically-constrained pair of one-forms satisfying some first-order PDEs. **[Cortés, Lazaroiu, Shahbazi '19]**.

Parallel spinor flows

Assume from now on that (M, g) is globally hyperbolic. As before, write:

$$g = -\beta_t^2 \mathrm{d}t^2 + h_t \,, \qquad t \in \mathcal{I} \,.$$

Let $\mathfrak{e}^t = (e_u^t, e_l^t, e_n^t)$ be a **family of orthonormal coframes** on Σ for $\{h_t\}_{t\in\mathcal{I}}$.

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(M, g) **admits a real parallel spinor** *if and only if:*

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$$\mathrm{d}\mathfrak{e}^t = \Theta_t(\mathfrak{e}^t) \wedge e^t_u \,, \quad [\Theta_t(e^t_u)] = 0 \in H^1(\Sigma, \mathbb{R}) \,,$$

where $a = u, l, n$, $h_t = e^t_u \otimes e^t_u + e^t_l \otimes e^t_l + e^t_n \otimes e^t_n$ *and* $\Theta_t = -\frac{1}{2\beta_t} \partial_t h_t$,

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These are the **parallel spinor flow equations**. Their solutions are families $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ of functions and coframes on Σ called **parallel spinor flows**.

Ricci curvature of (M, g) with a real parallel spinor

Proposition

Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ *be a parallel spinor flow. The Ricci curvature of* (M, g) *reads:*

$$\mathrm{Ric}^g = \frac{1}{2}\mathcal{H}_t e^{-2\mathfrak{f}_t} u \otimes u\,, \qquad u = e^{\mathfrak{f}_t}(\beta_t \mathrm{d}t + e^t_u)$$

where u *is the parallel one-form associated to a real parallel spinor,* $\mathcal{H}_t = \mathrm{Scal}^{h_t} - |\Theta_t|^2_{h_t} + \mathrm{Tr}_{h_t}(\Theta_t)^2$ *and* $\{\mathfrak{f}_t\}_{t\in\mathcal{I}}$ *is a family of functions satisfying* $\mathrm{d}\mathfrak{f}_t = -\Theta_t(e^t_u)$ *and* $\partial_t\mathfrak{f}_t = \mathrm{d}\beta_t(e^t_u)$.

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It is well known that Ric^g for a Lorentzian manifold admitting a real parallel spinor takes the form $\mathrm{Ric}^g = f\, u \otimes u$ for some $f \in C^\infty(M)$. To the best of my knowledge, this is the **first characterization** of such f in the case of globally hyperbolic manifolds.

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Let us analyze the parallel spinor flow equations:

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- The last equation of the parallel spinor flow is a **cohomological condition**, equivalent to $\mathrm{d}(\Theta_t(e^t_u)) = 0$ for simply connected Σ. However, it may restrict the discrete quotients to which a given solution on the universal cover descends.

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- The last equation of the parallel spinor flow is a **cohomological condition**, equivalent to $\mathrm{d}(\Theta_t(e^t_u)) = 0$ for simply connected Σ. However, it may restrict the discrete quotients to which a given solution on the universal cover descends.
- The **first two** equations are clearly **evolution equations**, since they have explicit time derivatives. The **second two equations** only contain derivatives along Σ and, evaluated at $t = 0$, **yield constraint equations**.

Constraint equations. Parallel Cauchy differential system

By the previous analysis and since the problem of a real parallel spinor is well-posed **[Lischewski '15; Leistner, Lischewski '19]**:

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The previous system of equations on Σ is called the **parallel Cauchy differential system**. Its solutions $(\mathfrak{e}, \Theta)$ are called **parallel Cauchy pairs**.

Parallel Cauchy differential system and GR constraint equations

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Since from $(\mathfrak{e}, \Theta)$ we may obtain an initial GR data set $(h_{\mathfrak{e}}, \Theta)$, we may wonder: Is it possible to have **parallel Cauchy pairs** which satisfy **additionally** the **GR constraint equations**?

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Answer: Yes! In fact, a **parallel Cauchy pair** $(\mathfrak{e}, \Theta)$ fulfills the **GR constraint equations** if and only if the **Hamiltonian constraint is satisfied**, and explicit examples may be constructed. This naturally prompts us the following question:

Do the **parallel spinor** and the **Einstein flows coincide** on **common initial data**?

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The **parallel spinor flow preserves** *the* **vacuum momentum** *and* **Hamiltonian constraints**.

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1. *A* **solution** (Σ, h, Θ) *to the* **vacuum GR constraint equations** *produces a Ricci flat Lorentzian development carrying a* **real parallel spinor** *if and only if there exists a global orthonormal coframe* $\mathfrak{e}$ *such that* $(\mathfrak{e}, \Theta)$ *is a* **parallel Cauchy pair**.

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2. (M, g) *with a* **real parallel spinor** *is* **Ricci flat** *if and only if there exists an adapted Cauchy surface* $\Sigma \subset M$ *whose* **Hamiltonian constraint vanishes**.

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This poses the **intriguing possibility** of using these **first-order spinorial flows** to construct special GR solutions, whose equations are of second order in derivatives.

Left-invariant parallel spinor flows

Let G be a connected and simply connected 3-dimensional Lie group. A **parallel spinor flow** $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ on G is **left-invariant** if both β_t and $\mathfrak{e}^t$ are left-invariant for every $t \in \mathcal{I}$. This implies h_t is left-invariant metric for every t and $\beta_t \in C^\infty(\mathcal{I})$ is constant for each t.

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The subsequent **left-invariant parallel spinor flow equations** are:

$$\partial_t \mathfrak{e}^t + \beta_t \Theta^t(\mathfrak{e}^t) = 0\,, \quad \mathrm{d}\mathfrak{e}^t = \Theta^t(\mathfrak{e}^t) \wedge e^t_u\,, \quad \partial_t(\Theta^t(e^t_u)) = 0\,, \quad \mathrm{d}\Theta^t(e^t_u) = 0\,,$$

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Let G be a connected and simply connected 3-dimensional Lie group. A **parallel spinor flow** $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ on G is **left-invariant** if both β_t and $\mathfrak{e}^t$ are left-invariant for every $t \in \mathcal{I}$. This implies h_t is left-invariant metric for every t and $\beta_t \in C^\infty(\mathcal{I})$ is constant for each t.

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We have been able to **classify** all left-invariant parallel spinor flows. For the sake of concreteness, we just show here an explicit example of solution.

Example of left-invariant parallel spinor flow

Let (G, h) be a connected and simply connected 3-dimensional Riemannian Lie group admitting an orthonormal left-invariant coframe (e_u, e_l, e_n) such that:

$$\mathrm{d}\mathfrak{e} = \Theta(\mathfrak{e}) \wedge e_u\,, \quad \Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ll} e_l \otimes e_l\,, \quad \Theta_{uu}, \Theta_{ll} \in \mathbb{R}\backslash 0\,.$$

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$$e^t_u = (1 - \Theta_{uu}\mathcal{B}_t) e_u \,, \quad e^t_l = (1 - \Theta_{uu}\mathcal{B}_t)^{\Theta_{ll}/\Theta_{uu}} e_l \,, \quad e^t_n = e_n \,, \qquad \mathcal{B}_t = \int_0^t \beta_\sigma \mathrm{d}\sigma \,.$$

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Then $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ defines a **left-invariant parallel spinor flow** on G for every $\{\beta_t\}_{t\in\mathcal{I}}$ and $t \in \mathcal{I}$ such that $1 - \Theta_{uu}\mathcal{B}_t > 0$.

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$$\mathrm{Ric}^g = \frac{\mathcal{H}}{2(1-\Theta_{uu}\mathcal{B}_t)^2}(\beta_t \mathrm{d}t + e^t_u) \otimes (\beta_t \mathrm{d}t + e^t_u)\,,$$

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Conclusions

1. The **Cauchy problem of pp-waves** (equivalently, of real parallel spinors) can be rephrased in terms of a system of **partial differential equations** of **first order** in time for a family of orthonormal coframes and functions on the Cauchy surface.

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2. The **parallel spinor flow** and the **vacuum Einstein flow coincide** on common initial data.

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太感谢了！

Thank you so much!