Uniqueness of supersymmetric AdS_5 black holes

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Contents

Black holes in holography

AdS/CFT

asymptotically AdS_D solutions of Einstein equations	$\leftrightarrow$	states of the dual CFT_{D-1} on the asymptotic boundary (conformal Minkowski)

Black holes are particularly important: they are dual to states with entropy.

- For $D \geq 4$ we are far from having complete understanding of the duality.
- On the gravity side, because of cosmological constant, standard integrability techniques like Lax pair, sigma model, etc., are inapplicable.
- Consequently, there are few results on the gravitational side: BHs are not classified.
- On the contrast, there is a recent CFT derivation of the entropy of a *known* **supersymmetric** $D = 5$ black hole. [2018 Cabo-Bizet, Cassani, Martelli, Murthy]

5d minimal gauged supergravity

We consider $5d$ dimensional minimal gauged supergravity as the simplest higher dimensional model:

- minimal matter content: Einstein-Maxwell-Chern-Simons with $\Lambda = -12/\ell^2$,
- consistent truncation of $D = 10$ supergravity, and is compatible with supersymmetry.

The action is

$$S = \frac{1}{4\pi G} \int \left(\frac{1}{4} (R_5 + 12/\ell^2) \star 1 - \frac{1}{2} F \wedge \star F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right) \qquad (1)$$

where ℓ is AdS length and $F = dA$ – field tensor.

5d minimal gauged supergravity. Solutions

There are two known families of smooth supersymmetric black holes:

Examples

- Gutowski-Reall (GR) black hole: 1-parametric solution with $SU(2) \times U(1) \times \mathbb{R}_t$ symmetry
- Chong-Cvetic-Lu-Pope (CCLP) black hole: 2-parametric solution with $U(1)^2 \times \mathbb{R}_t$ symmetry (contains GR)

Besides that not much progress was made:

- No classification, no black hole theorems.
- Asymptotically flat supergravity allows for the whole zoo: multi-black holes, black lenses, etc. [2014 Kunduri, Lucietti, 2022 Katona, Lucietti]
- Solutions with only $U(1) \times \mathbb{R}_t$ symmetry are not ruled out (no enhanced rigidity theorem).

- Black rings **are** ruled out. [2006 Kunduri, Lucietti, Reall]
- **Near-horizon geometries** are totally classified. [2006 KLR, 2013 Grover, Gutowski, Papadopoulos, Sabra]

Results. Physics

We prove the first uniqueness theorem for black holes with cosmological constant in dimensions $D > 3$.

Black hole uniqueness theorem

Any supersymmetric and $SU(2)$-symmetric solution to five-dimensional minimal gauged supergravity, that is timelike outside an analytic horizon with compact cross-sections, must be a Gutowski-Reall black hole or its near-horizon geometry.

More details can be found in [2105.08542 Lucietti, SO]

Results. Geometry

Local classification of SU(2) Kähler spaces

The most general Kähler metric with a cohomogeneity-1 $SU(2)$ symmetry can be written in the frame

$$h = \mathrm{d}\rho^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{2}$$

where σ_i are $SU(2)$ right-invariant 1-forms, and metric functions satisfy

$$c\,T' = 1 - T^2\,, \qquad T := \frac{b}{a} \tag{3}$$

$$c = (ab)'\,. \tag{4}$$

The Kähler form is simply $\Omega^3 = \mathrm{d}(ab\,\sigma_3)$.

Note that (2) could have crossterms $\sigma_i\sigma_j$, $i \neq j$!!

Corollary: Symmetry enhancement

Let a, b, c are all positive and C^1 for $\rho > \rho_0$, $(abc)|_{\rho_0} = 0$, and $\lim_{\rho\to\rho_0^+} T \neq \infty$. If $T|_{\rho_0} = 1$ then $a = b$ everywhere, i.e. the metric has $SU(2)\times U(1)$ symmetry.

Local form of solution and Kähler geometry

The solution (5d metric and gauge field) are reconstructed from metric and curvature of relevant Kähler geometry. This geometry is bound by a constraint. [2003 Gauntlett, Gutowski]

In particular, they show:

- Supersymmetry demands existence of causal Killing vector field V and a 2-form Ω^3.
- The metric can be locally decomposed with respect to timelike V:

$$g_5 = -f^2(\mathrm{d}t + \omega)^2 + f^{-1}h \tag{5}$$

- where $V = \frac{\partial}{\partial t}$, $V^2 = -f^2$ and (h, Ω^3) are the Kähler geometry, i.e., Ω *is ASD covariantly constant 2-form on* h, *and* $\Omega^3{}_m{}^k \Omega^3{}_k{}^n = -\delta^m_n$.
- (f, ω, h, Ω^3) and gauge field $F = \mathrm{d}A$ are invariant under Killing field V.

Reconstructing solution from Kähler data

The solution (5d metric and gauge field) are reconstructed from metric and curvature of relevant Kähler geometry (h, Ω^3). This geometry is bound by a constraint. [2003 Gauntlett, Gutowski]

Data reconstruction (1)

- The metric is constructed as $g_5 = -f^2(\mathrm{d}t + \omega)^2 + f^{-1}h$
- The field strength is constructed out of f, ω, Kähler form and curvature
- The f is the inverse of scalar curvature: $f = -\frac{24}{\ell^2}\frac{1}{R}$

Data reconstruction (2)

Solving for ω gives the constraint on curvature of (h, Ω^3):

$$\mathrm{d}\omega = \frac{\ell^3}{48}\left[R\left(\mathcal{R} - \frac{R}{4}\Omega^3\right) - \left(\lambda_1\Omega^1 + \lambda_2\Omega^2 + \lambda_3\Omega^3\right)\right] \tag{6}$$

where $\mathcal{R}$ is the Ricci form and $\Omega^1, \Omega^2, \Omega^3$ are a basis of ASD 2-forms.
For general h the (6) is not integrable: $\mathrm{d}^2\omega \neq 0$!!
[2015 Cassani, Martelli et al.]

Kähler geometry and black holes

Black hole uniqueness proof strategy

- *Classify Kähler spaces with* $SU(2)$ *isometry. If there is a point where* $a = b$ *then* $a = b$ *everywhere, and solution depends on one function.*
- Show that such point always exists for any smooth black hole with $SU(2)$ isometry — in fact, this is the horizon!
- Obtain the supersymmetric constraint $\mathrm{d}^2\omega = 0$ explicitly, and show that there is a unique analytic solution to this equation with black hole boundary conditions.

SU(2) symmetry

$SU(2)$ isometry

- $SU(2)$ is the isometry of (M, g_5) with 3d orbits,
- $SU(2)$ commutes with supersymmetry: the full isometry is $SU(2) \times \mathbb{R}_t$,
- Maxwell field F is $SU(2)$-invariant as well;
- Consequently, (h, Ω^3) inherits $SU(2)$ isometry.

Example

Gutowski-Reall black hole

$$h = \mathrm{d}\rho^2 + \alpha^2 \ell^2 \sinh(\rho/\ell)^2 (\sigma_1^2 + \sigma_2^2) + \alpha^4 \ell^2 \sinh(2\rho/\ell) \sigma_3^2 \, , \tag{7}$$

$$\Omega^3 = \mathrm{d}\left(\alpha^2 \ell^2 \sinh(\rho/\ell)^2 \sigma_3\right) \tag{8}$$

where $\alpha > \frac{1}{2}$ corresponds to black hole and $\alpha = \frac{1}{2}$ is AdS_5. The horizon is at $\rho = 0$ where h becomes singular.

Kähler spaces with SU(2) isometry

General $SU(2)$-invariant Kähler geometry

- Choose orthonormal frame $e^0 = \mathrm{d}\rho$, $e^i = E^i_j(\rho)\sigma_j$:

$$h = (e^0)^2 + e^i e^i, \qquad \Omega^i = e^0 \wedge e^i - \frac{1}{2}\epsilon_{ijk} e^j \wedge e^k \ . \tag{9}$$

- Using $e^i \to O^i_j(\rho) e^j$ we can pick $\Omega = \Omega^3$.
- Parameterize the frame by functions of ρ *(where $abc \not\equiv 0$)*

$$e^0 = \mathrm{d}\rho, \quad e^1 = a\sigma_1 + b_1\sigma_2 + c_1\sigma_3, \quad e^2 = b\sigma_2 + c_2\sigma_3, \quad e^3 = c\sigma_3, \tag{10}$$

$$\text{hence,} \quad \Omega^3 = c\,\mathrm{d}\rho \wedge \sigma_3 - ab\ \sigma_1 \wedge \sigma_2 + ac_2\ \sigma_1 \wedge \sigma_2 - (b_1 c_2 - b c_1)\sigma_2 \wedge \sigma_3. \tag{11}$$

Resolving the Kähler geometry (1)

- By construction (9), Ω^3 is ASD almost-complex structure on h.
- Closure of Ω^3 is equivalent to

$$c = (ab)', \quad c_1 = c_2 = 0. \tag{12}$$

Similarly, for Kähler-Einstein spaces it was done in [1993 Dancer, Strachan]

Geometric result

Resolving the Kähler geometry (2)

- The integrability of complex structure further demands

$$2bca' = b^2 - a^2 + c^2 - b_1^2, \qquad 2(abc)b_1' = -b_1(a^2 + 3b^2 - c^2 + b_1^2). \quad (13)$$

- Using constant rotation $\sigma_i \to R_{ij}\sigma_j$ we can always bring $b_1(\rho_0) = 0$ at some point ρ_0. Then by ODE uniqueness theorem we have $b_1 \equiv 0$.
- Altogether the constraints can be rewritten as

$$c\,T' = 1 - T^2, \qquad c = (ab)', \qquad b_1 = c_1 = c_2 = 0, \qquad T := \frac{b}{a}. \quad (14)$$

Corollary: Symmetry enhancement

Let a, b, c are all positive and C^1 for $\rho > \rho_0$, $(abc)|_{\rho_0} = 0$, and $\lim_{\rho\to\rho_0^+} T \neq \infty$. If $T|_{\rho_0} = 1$ then $a = b$ everywhere, i.e. the metric has $SU(2)\times U(1)$ symmetry.

Kähler geometry and black holes

Black hole uniqueness proof strategy

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- *Show that such point always exists for any smooth black hole with* $SU(2)$ *isometry — in fact, this is the horizon!*
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Near-horizon geometry

Gaussian null coordinates

- The consistent way to locally describe a spacetime with horizon is to introduce a particular chart, Gaussian null coordinates:

$$g = -\lambda^2\Delta^2(\lambda)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}\lambda + 2\lambda h_i(\lambda)\hat{\sigma}_i\mathrm{d}v + \gamma_{ij}(\lambda)\hat{\sigma}_i\hat{\sigma}_j \ , \qquad (15)$$

  where we demanded that $SU(2)$ is the isometry of the spacetime (M, g).
- The horizon is at $\lambda = 0$, and ∂_λ are null geodesics transverse to it.
- $V = \partial_v$ is the supersymmetric Killing field, and $\hat{\sigma}_i$ are right-invariant 1-forms.
- There is an important scaling limit $(v, \lambda) \to (v/\epsilon, \epsilon\lambda)$.
  The limit $\epsilon \to 0$ is the **near-horizon geometry**. Near-horizon geometries are algebraically special.

Near-horizon geometries with compact horizon cross-sections have been completely classified in this theory [2006 KLR, 2013 Grover,Gutowski, Papadopoulos, Sabra]:
The unique regular near-horizon geometry is that of the CCLP (or GR if we have $SU(2)$ symmetry).

Matching to NH (1)

- The near-horizon geometry of GR is the only one with $SU(2)$ isometry in this theory.
- We need to relate the GNC $(v, \lambda, \hat{\sigma}_i)$ and (t, ρ, σ_i) charts.

Matching

- The supersymmetric Killing field in both charts is $V = \partial_v = \partial_t$.
- Consequently, the metric h on the orthogonal base is invariantly defined. In GNC chart

$$h = \left(\frac{\Delta}{\Delta^2 + h_i h^i}\right) \frac{\mathrm{d}\lambda^2}{\lambda} + \lambda \Delta q_{ij} \left(\hat{\sigma}_i + \frac{k^i \mathrm{d}\lambda}{\lambda}\right) \left(\hat{\sigma}_j + \frac{k^j \mathrm{d}\lambda}{\lambda}\right) , \tag{16}$$

for $\lambda > 0$, $h^i = \gamma^{ij} h_j$ and

$$q_{ij} := \gamma_{ij} + \frac{h_i h_j}{\Delta^2}, \qquad k^i := \frac{h^i}{\Delta^2 + h^i h_i}. \tag{17}$$

Matching to NH (2)

Matching

- The SU(2) right invariant 1-forms are defined up to

$$\sigma_i := A_{ij}(\lambda)(\hat{\sigma}_j + B_j(\lambda)\mathrm{d}\lambda)\ , \tag{18}$$

$$\text{where} \qquad A'_{ij} = A_{ik}C_{kj}, \qquad C_{ij} := \epsilon_{ijk}\frac{k^i}{\lambda}\ . \tag{19}$$

- Finally, by the nature of $SU(2)$ symmetry, the radial coordinates are related as

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\lambda}\right)^2 = \left(\frac{\Delta}{\Delta^2 + h_i h^i}\right)\frac{1}{\lambda}\ , \tag{20}$$

This implies that λ is an even function of ρ.
Consequently, all metric functions must be smooth in λ, they are even functions in terms of ρ.

Matching to NH (3)

Proposition

For a timelike supersymmetric $SU(2)$-isometric solution to this theory which has a smooth horizon, the horizon corresponds to a conical singularity $\rho = 0$ of Kähler base and the metric functions a, b, c in the chart (ρ, σ_i) behave at the horizon as

- $a^2 = b^2 = \alpha^2\rho^2 + O(\rho^4)$ and $c^2 = 4\alpha^2\rho^2 + O(\rho^4)$,
- a, b, c are even functions of ρ at the horizon.

As a consequence, we have $a^2 = b^2$ everywhere, i.e. the full solution has enhanced $SU(2) \times U(1)$ symmetry.

Kähler geometry and black holes

Black hole uniqueness proof strategy

- Classify Kähler spaces with $SU(2)$ isometry. If there is a point where $a = b$ then $a = b$ everywhere, and solution depends on one function.
- Show that such point always exists for any smooth black hole with $SU(2)$ isometry — in fact, this is the horizon!
- *Obtain the supersymmetric constraint* $d^2\omega = 0$ *explicitly, and show that there is a unique analytic solution to this equation with black hole boundary conditions.*

Supersymmetric constraint

New coordinate

It is convenient to introduce (slightly) different radial coordinate: $r := 2a(\rho)$. Then the Kähler space is parameterized by one function $V(r) := 4a'(\rho)^2$

$$h = \mathrm{d}\rho^2 + a^2(\sigma_1^2 + \sigma_2^2) + c^2\sigma_3^2 = \frac{\mathrm{d}r^2}{V(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{r^2 V(r)}{4}\sigma_3^2 \tag{21}$$

- The boundary conditions from near-horizon geometry translate as $V(r = 0) = 4\alpha^2 \geq 1$; the horizon is at $r = 0$.
- $V(r)$ **is an even function of** r **at the horizon.**

Example

In this chart is that GR black hole is a quadratic function of r:

$$V_{GR}(r) = 4\alpha^2 + r^2/\ell^2. \tag{22}$$

This means that the series expansions in r **truncate at the second order.**

Supersymmetric constraint. Series expansions

Supersymmetric constraint

The constraint $\mathrm{d}^2\omega = 0$ can be calculated explicitly in this geometry.
It is equivalent to the following ODE

$$3r^4VV^{(5)} + 6r^4V'V^{(4)} + 30r^3VV^{(4)} + 44r^3V'V^{(3)} + r^2(47V-32)V^{(3)}$$
$$+8r^3(V'')^2 - 3r(13V+32)V'' + 26r^2V'V'' - 34rV'^2 + 3V'(13V+32) = 0$$

Series expansions

Introducing $V(r) = \sum \frac{V_n r^n}{n!}$ where $V_0 = 4\alpha^2 > 1$ we have

- Orders r^0 and r^1 give $V_1 = 0$ while V_2 is a free parameter.
- Order r^2 allows for two branches: $V_3\left(V_0 - \frac{32}{11}\right) = 0$.
  Smooth black holes are even functions of r, hence $V_3 = 0$.
  $V_3 \neq 0$ corresponds to squashed S^3 black hole [2017 Kunz, Radu et al.]
- Order r^3 gives $V_4 = 0$ for black holes.
- **Induction argument for $n \geq 4$.**

The uniqueness theorem. Proof

Induction argument

- If $V(r) = V_0 + \frac{V_2 r^2}{2} + \frac{r^n V_n}{n!} + O(r^{n+1})$ for $n \geq 4$, then each order of ODE looks like

$$\frac{[3V_0(n^2-16)+32(V_0-1)](n^2-4)V_n r^{n-1}}{(n-1)!} + O(r^n) = 0\ . \tag{23}$$

- Hence, $V_n = 0\ \forall n \geq 4$. The series truncate and converge.
- If $V_2 = 0$ we recover near-horizon geometry with $V_{NH} = 4\alpha^2$.
- Else, the form $V_{GR} = 4\alpha^2 + r^2/\ell^2$ is recovered by parameterizing $V_2 = 2/\ell^2$.

Black hole uniqueness theorem

Any supersymmetric and $SU(2)$-symmetric solution to five-dimensional minimal gauged supergravity, that is timelike outside an analytic horizon with compact cross-sections, must be a Gutowski-Reall black hole or its near-horizon geometry.

Other solutions with $SU(2)$ symmetry

$SU(2)$ case

We have a complicated 5^{th} order ODE that admits a simple solution.

- *Can we further simplify the ODE?*
- *Is smoothness enough for uniqueness?*
- *What is the general solution to the ODE?*

Example

We have rediscovered a smooth soliton with $\mathrm{AdS}_5/\mathbb{Z}_p$ asymptotic

$$V = c_0 + \frac{r^2}{\ell^2} + \frac{c_2}{r^2} + \frac{c_4}{r^4}, \qquad c_2^2 = 3(c_0 - 1)c_4 \ . \tag{24}$$

Its 5d metric has a removable conical singularity at a bolt.

$SU(2)$ ODE simplification

$SU(2)$ case

We have a complicated 5^{th} order ODE that admits a simple solution.

- *Can we further simplify the ODE?*
- *Is smoothness enough for uniqueness?*
- *What is the general solution to the ODE?*

ODE simplification

The 5^{th} order ODE has a significant hidden symmetry, and can be reduced to 2^{nd} order by introduction of new function $\mu(V) = \dfrac{dV}{dt} = r\dfrac{dV}{dr}$:

$$2\mu^3\left(8 + 9V\mu''\right) + \mu^2\left(9V\mu'^2 - 84V - 96\right) = -64(V^3 + 6V^2 + \kappa_1 V + \kappa_0) \quad (25)$$

where $\kappa_{0,1}$ are integration constants.
For smooth black holes, boundary conditions read as

$$\mu|_{V_0} = 0\,, \qquad \mu'|_{V_0} = 2\,, \qquad V_0 = 4\alpha^2. \quad (26)$$

General solution is not known!

Discussion. Generalizations

- We did not do any assumptions on the asymptotics, yet we rule out locally asymptotically AdS_5 solutions in this class.
- In the context of supergravity we are interested in non-compact Kähler spaces with conical singularities.
- General solution to the ODE is not known!

Generalization to other symmetry classes?

- What about $U(1)^2$? [To appear Lucietti, Ntokos, SO].
- Can we say something about just $U(1)$? Is there a similar symmetry enhancement to $U(1)^2$?