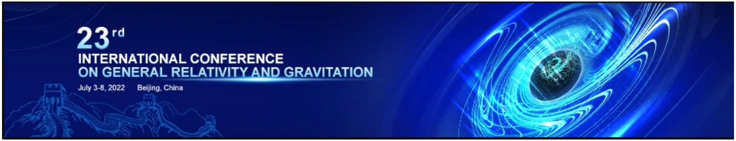
# Linear instability of gravitational and electromagnetic perturbations of Extreme Reissner–Nordström

Marios Antonios Apetroaie

University of Toronto

July 8, 2022



# The Stability Problem

The study of evolution of perturbed stationary black holes, as solutions to the Einstein equations.

**Stability**

Given a Cauchy hypersurface, $\Sigma$, of a stationary solution $(\mathcal{M}, g_{\mu\nu})$ arising from initial data $(\Sigma, \bar{g}, K)^{(1)}$, let $(\Sigma', \bar{g}', K')$ be admissible initial data sufficiently *close* to $^{(1)}$ and consider its maximal Cauchy development.

# The Stability Problem

The study of evolution of perturbed stationary black holes, as solutions to the Einstein equations.

## Stability

Given a Cauchy hypersurface, $\Sigma$, of a stationary solution $(\mathcal{M}, g_{\mu\nu})$ arising from initial data $(\Sigma, \bar{g}, K)^{(1)}$, let $(\Sigma', \bar{g}', K')$ be admissible initial data sufficiently *close* to $^{(1)}$ and consider its maximal Cauchy development.

We are interested in the underlying properties of this development relative to the background solution and what is its asymptotic behavior.

# Intermediate problems to consider

- Linear scalar wave equation
- Linear stability
- Full non-linear stability

The study of scalar wave equation on a fixed background spacetime $(\mathcal{M}, g)$

$$\Box_g \psi = 0$$

## Contributors

Dafermos, Rodnianski, Andersson, Tataru, Moschidis, Blue, Holzegel, Sbierski, Shlapentokh-Rothman, Dyatlov, Häfner, Bony, Smulevici, Klainerman, Ionescu, Tohaneanu, Sterbenz, Soffer, Schlue, Luk, Oh, Finster, Kamran, Smoller, Yau, Donninger, Schlag, Vasy, Hintz, Metcalfe, Wald, Franzen, Teixeira da Costa, ...

# Intermediate problems to consider

- Linear scalar wave equation
- Linear stability
- Full non-linear stability

The study of linearized gravity, i.e. we consider linear perturbations of the Einstein equations around a specific black hole solution.

## Contributors

- **Schwarzschild:** Blue (2008), Pasqualotto (2016), Dafermos-Holzegel-Rodnianski (2016), Keller-Hung-Wang (2017), Johnson(2018), Hung (2018) ...
- **Reissner-Nordström/Kerr-Newman:** Elena Giorgi (2018, 2019, 2020) ..
- **Kerr:** Dafermos-Holzegel-Rodnianski (2017), Ma (2017), Andersson-Bäckdahl-Blue-Ma (2019), Häfner-Hintz-Vasy (2019), Shlapentokh-Rothman-Teixeira da Costa (2020)

# Intermediate problems to consider

- Linear scalar wave equation
- Linear stability
- Full non-linear stability

Study the full non-linear Einstein equations with no assumption on the form of perturbations.

**Contributors**

- **Minkowski:** Christodoulou-Klainerman (1993), Lindblad-Rodnianski 2004, Bieri (2009)
- **Schwarzschild:** Klainerman-Szeftel (2017), Dafermos-Holzegel-Rodnianski-Taylor (2021)
- **Kerr:** Klainerman-Szeftel (2019, 2021), Giorgi-Klainerman-Szeftel (2022)

# Motivation

The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

Elena Giorgi

Department of Mathematics, Princeton University, Princeton, USA. E-mail: egiorgi@princeton.edu

Received: 19 October 2019 / Accepted: 2 September 2020

# Motivation

**The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$**

**Elena Giorgi**

Department of Mathematics, Princeton University, Princeton, USA. E-mail: egiorgi@princeton.edu

Received: 19 October 2019 / Accepted: 2 September 2020


## Questions

- What about the linear stability of the **Extremal** Reissner-Nordström spacetime, $|Q| = M$

# Motivation

**The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$**

**Elena Giorgi**

Department of Mathematics, Princeton University, Princeton, USA. E-mail: egiorgi@princeton.edu

Received: 19 October 2019 / Accepted: 2 September 2020


## Questions

- What about the linear stability of the **Extremal** Reissner-Nordström spacetime, $|\mathcal{Q}| = M$
- What is it known for the scalar wave equation in ERN background?

# Relevant qualitative results on ERN

Let $\psi$ be a solution to the homogeneous wave equation with respect to the ERN metric

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

**Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010 )**

For all solutions $\psi_\ell$ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, the quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i(M,\ell) \cdot \partial_r^i \psi_\ell$$

is conserved along the null geodesics of $\mathcal{H}^+$.

# Relevant qualitative results on ERN

Let $\psi$ be a solution to the homogeneous wave equation with respect to the ERN metric

$$\Box_{g_{ERN}} \psi = 0 \quad (1)$$

**Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010 )**

For all solutions $\psi_\ell$ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, the quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i(M,\ell) \cdot \partial_r^i \psi_\ell$$

is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, almost everywhere on $\mathcal{H}^+$, for generic initial data of $\psi$. )

# Relevant qualitative results on ERN

Let $\psi$ be a solution to the homogeneous wave equation with respect to the ERN metric

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

**Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010 )**

For all solutions $\psi_\ell$ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, the quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i(M,\ell) \cdot \partial_r^i \psi_\ell$$

is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, almost everywhere on $\mathcal{H}^+$, for generic initial data of $\psi$. )

- **Decay:** For all $k \leq \ell$, we have $\left|\partial_r^k \psi_\ell\right| \xrightarrow{\tau\to\infty} 0$

# Relevant qualitative results on ERN

Let $\psi$ be a solution to the homogeneous wave equation with respect to the ERN metric

$$\Box_{g_{ERN}} \psi = 0 \tag{1}$$

**Conservation Laws along $\mathcal{H}^+$ (Stefanos Aretakis.2010 )**

For all solutions $\psi_\ell$ to (1), supported on the **fixed** angular frequency $\ell \in \mathbb{N}$, the quantity

$$H_\ell[\psi_\ell] = \partial_r^{\ell+1}\psi_\ell + \sum_{i=0}^{\ell} \beta_i(M,\ell) \cdot \partial_r^i \psi_\ell$$

is conserved along the null geodesics of $\mathcal{H}^+$. ($H_\ell[\psi] \neq 0$, almost everywhere on $\mathcal{H}^+$, for generic initial data of $\psi$. )

- **Decay:** For all $k \leq \ell$, we have $\left|\partial_r^k \psi_\ell\right| \xrightarrow{\tau \to \infty} 0$
- **Blow up:** For all $k \in \mathbb{N}$, we have $\left|\partial_r^{\ell+1+k} \psi_\ell\right| \gtrsim H_\ell \cdot \tau^k$

# Linearized gravity

A first step in studying the linear stability is to identify the so called:

**Gauge-Invariant Quantities**

Quantities that remain invariant under linear perturbations corresponding to smooth coordinate transformation of the background metric.

These quantities govern the gravitational and electromagnetic perturbations, unlocking the control of **gauge dependent** quantities as well.

# Linearized gravity

A first step in studying the linear stability is to identify the so called:

**Gauge-Invariant Quantities**

Quantities that remain invariant under linear perturbations corresponding to smooth coordinate transformation of the background metric.

These quantities govern the gravitational and electromagnetic perturbations, unlocking the control of **gauge dependent** quantities as well.

In view of expected instabilities along the event horizon $\mathcal{H}^+$ we restrict our attention to the study of gauge-invariant components alone.

# Linearized gravity

A first step in studying the linear stability is to identify the so called:

**Gauge-Invariant Quantities**

Quantities that remain invariant under linear perturbations corresponding to smooth coordinate transformation of the background metric. $(\star)$

These quantities govern the gravitational and electromagnetic perturbations, unlocking the control of **gauge dependent** quantities as well.

In view of expected instabilities along the event horizon $\mathcal{H}^+$ we restrict our attention to the study of gauge-invariant components alone.

$$(\star) \begin{cases} \text{2-tensors: } \alpha,\ \mathfrak{f} \\ \text{1-tensor: } \mathfrak{b} \end{cases}$$

# Linearized gravity

A first step in studying the linear stability is to identify the so called:

**Gauge-Invariant Quantities**

Quantities that remain invariant under linear perturbations corresponding to smooth coordinate transformation of the background metric. $(\star)$

These quantities govern the gravitational and electromagnetic perturbations, unlocking the control of **gauge dependent** quantities as well.

In view of expected instabilities along the event horizon $\mathcal{H}^+$ we restrict our attention to the study of gauge-invariant components alone.

$$(\star) \begin{cases} \text{2-tensors:} \quad \alpha, \ \mathfrak{f} \\ \text{1-tensor:} \quad \mathfrak{b} \end{cases}$$

⚠ The quantities $\mathfrak{f}$, $\mathfrak{b}$ are defined in terms of **both** electromagnetic **and** gravitational coefficients.

# Teukolsky system

- Ingoing Eddington-Finkelstein coordinates: $(\upsilon, r, \theta, \phi)$

$$g_{M,Q} = -D(r)d\upsilon^2 + 2d\upsilon dr + r^2 \cdot \gamma, \qquad D(r) := \left(1 - \frac{M}{r}\right)^2$$

- Teukolsky type operator

$$\mathcal{T}(\alpha) := \Box_{g_{ERN}} \alpha + c_1(r)\nabla_{\partial_\upsilon} \alpha + c_2(r)\nabla_{\partial_r} \alpha + V(r)\alpha$$

The gauge invariant quantities introduced earlier satisfy **coupled** generalized Teukolsky type equations

$$\mathcal{T}(\mathfrak{f}) = d_1(r) \cdot \nabla \widehat{\otimes} \mathfrak{b} + \ldots$$
$$\mathcal{T}(\mathfrak{b}) = d_2(r) \cdot div \mathfrak{f} + \ldots$$

Studying the above system directly is not possible. There is, however, a set of transformations that yield Regge-Wheeler type equations.

# Regge-Wheeler System

With a transformation of the following type

$$\begin{pmatrix} \boldsymbol{q}^F \\ \boldsymbol{p} \end{pmatrix} = r \nabla\!\!\!/_{\partial_r} \begin{pmatrix} c_1(r) \cdot \mathfrak{f} \\ c_2(r) \cdot \mathfrak{b} \end{pmatrix}$$

we obtain

$$\Box_{g_{ERN}} \boldsymbol{q}^F - A_1(r) \cdot \boldsymbol{q}^F = h_1(r) \nabla \widehat{\otimes} \boldsymbol{p}$$
$$\Box_{g_{ERN}} \boldsymbol{p} - A_2(r) \cdot \boldsymbol{p} = h_2(r) div \boldsymbol{q}^F$$

# Regge-Wheeler System

With a transformation of the following type

$$\begin{pmatrix} \boldsymbol{q}^F \\ \boldsymbol{p} \end{pmatrix} = r \nabla\!\!\!/_{\partial_r} \begin{pmatrix} c_1(r) \cdot \mathfrak{f} \\ c_2(r) \cdot \mathfrak{b} \end{pmatrix}$$

we obtain

$$\Box_{g_{ERN}} \boldsymbol{q}^F - A_1(r) \cdot \boldsymbol{q}^F = h_1(r) \nabla \widehat{\otimes} \boldsymbol{p}$$

$$\Box_{g_{ERN}} \boldsymbol{p} - A_2(r) \cdot \boldsymbol{p} = h_2(r) div \, \boldsymbol{q}^F$$

$$\downarrow$$

$$\Box_{g_{ERN}} \phi + V_1(r)\phi = -\frac{1}{2r} \Delta\!\!\!/ \psi - \frac{1}{r^3}\psi$$

$$\Box_{g_{ERN}} \psi + V_2(r)\psi = \frac{8M^2}{r^3}\phi.$$

(Scalar Regge-Wheeler system)

# Regge-Wheeler System

With a transformation of the following type

$$\begin{pmatrix} \boldsymbol{q}^F \\ \boldsymbol{p} \end{pmatrix} = r\nabla\!\!\!/_{\partial_r}\begin{pmatrix} c_1(r)\cdot \mathfrak{f} \\ c_2(r)\cdot \mathfrak{b} \end{pmatrix}$$

we obtain

$$\Box_{g_{ERN}}\boldsymbol{q}^F - A_1(r)\cdot\boldsymbol{q}^F = h_1(r)\nabla\widehat{\otimes}\boldsymbol{p}$$
$$\Box_{g_{ERN}}\boldsymbol{p} - A_2(r)\cdot\boldsymbol{p} = h_2(r)div\,\boldsymbol{q}^F$$

$$\downarrow$$

$$\Box_{g_{ERN}}\phi + V_1(r)\phi = -\frac{1}{2r}\Delta\!\!\!/\psi - \frac{1}{r^3}\psi$$
$$\Box_{g_{ERN}}\psi + V_2(r)\psi = \frac{8M^2}{r^3}\phi.$$

(Scalar Regge-Wheeler system)

The scalars on the left are given by

$$\phi = r^2 d\!\!/iv(d\!\!/iv\,\boldsymbol{q}^F)$$
$$\psi = r\ d\!\!/iv(\boldsymbol{p}),$$

while for the potential functions we have

$$V_i(r) = \mathcal{O}\left(r^{-3}\right).$$

# Decoupling of Regge-Wheeler scalar system

Idea:

- Consider the spherical harmonics decomposition of the coupled scalar Regge-Wheeler system. (This way, only zeroth order terms appear)
  Note: $\phi_\ell$ is supported on frequencies $\ell \geq 2$, while $\psi_\ell$ on $\ell \geq 1$. Projecting on the $\ell = 1$ frequency the system naturally decouples.

# Decoupling of Regge-Wheeler scalar system

Idea:

- Consider the spherical harmonics decomposition of the coupled scalar Regge-Wheeler system. (This way, only zeroth order terms appear)
  Note: $\phi_\ell$ is supported on frequencies $\ell \geq 2$, while $\psi_\ell$ on $\ell \geq 1$. Projecting on the $\ell = 1$ frequency the system naturally decouples.
- Write the system as

$$\left(\Box_{g_{ERN}} + w(r)\right) \begin{pmatrix} \phi_\ell \\ \psi_\ell \end{pmatrix} = \boldsymbol{A}_{2\times 2} \begin{pmatrix} \phi_\ell \\ \psi_\ell \end{pmatrix}, \qquad \ell \geq 2$$

  where $\boldsymbol{A}_{2\times 2}$ is a **symmetric** $2 \times 2-$matrix.

# Decoupling of Regge-Wheeler scalar system

Idea:

- Consider the spherical harmonics decomposition of the coupled scalar Regge-Wheeler system. (This way, only zeroth order terms appear)
  Note: $\phi_\ell$ is supported on frequencies $\ell \geq 2$, while $\psi_\ell$ on $\ell \geq 1$. Projecting on the $\ell = 1$ frequency the system naturally decouples.
- Write the system as

$$\left(\Box_{g_{ERN}} + w(r)\right)\begin{pmatrix}\phi_\ell \\ \psi_\ell\end{pmatrix} = \boldsymbol{A}_{2\times 2}\begin{pmatrix}\phi_\ell \\ \psi_\ell\end{pmatrix}, \qquad \ell \geq 2$$

  where $\boldsymbol{A}_{2\times 2}$ is a **symmetric** $2 \times 2-$matrix.
- Diagonalize the system to obtain the decoupled equations

**Regge-Wheeler equations**

$$(RW) \qquad \left(\Box_{g_{ERN}} - V_i^{(\ell)}(r)\right)\Psi_i^{(\ell)} = 0, \qquad i \in \{1,2\}\,,\ \ell \geq i$$

where $\Psi_i^{(\ell)} = a_i(\ell, M)\phi_\ell + b_i(\ell, M)\psi_\ell$.
The potentials depend on $M, \ell$ and decay towards spacelike infinity like $V_i^{(\ell)}(r) = \mathcal{O}(\frac{1}{r^3})$.

# Qualitative behavior of solutions to (RW)

## Conservation Laws along $\mathcal{H}^+$ (Upcoming work)

Let $\ell \in \mathbb{N}$, with $\ell \geq i$, $i \in \{1,2\}$, then there exist constants $c_i^j$, $j = 0, 1, \ldots, \ell + (-1)^{i+1}$ depending on $M, \ell$, such that for all solution $\Psi_i^{(\ell)}$ to (RW), the quantities

$$
\begin{aligned}
H_\ell[\Psi_1] &= \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)} \\
H_\ell[\Psi_2] &= \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)}
\end{aligned}
\tag{2}
$$

are conserved along the null generators of $\mathcal{H}^+$.

# Qualitative behavior of solutions to (RW)

## Conservation Laws along $\mathcal{H}^+$ (Upcoming work)

Let $\ell \in \mathbb{N}$, with $\ell \geq i$, $i \in \{1,2\}$, then there exist constants $c_i^j$, $j = 0, 1, \ldots, \ell + (-1)^{i+1}$ depending on $M, \ell$, such that for all solution $\Psi_i^{(\ell)}$ to (RW), the quantities

$$\begin{aligned} H_\ell[\Psi_1] &= \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)} \\ H_\ell[\Psi_2] &= \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)} \end{aligned} \tag{2}$$

are conserved along the null generators of $\mathcal{H}^+$.

## Decay along $\mathcal{H}^+$

- $\left|\partial_r^k \Psi_i^{(\ell)}\right| \lesssim \tau^{-1}, \quad \forall\ k \leq (\ell-2)-(-1)^i$
- $\left|\partial_r^{\ell-1-(-1)^i} \Psi_i^{(\ell)}\right| \lesssim \tau^{-\frac{3}{4}}$
- $\left|\partial_r^{\ell-(-1)^i} \Psi_i^{(\ell)}\right| \lesssim \tau^{-\frac{1}{4}}$

# Qualitative behavior of solutions to (RW)

## Conservation Laws along $\mathcal{H}^+$ (Upcoming work)

Let $\ell \in \mathbb{N}$, with $\ell \geq i$, $i \in \{1, 2\}$, then there exist constants $c_i^j$, $j = 0, 1, \ldots, \ell + (-1)^{i+1}$ depending on $M, \ell$, such that for all solution $\Psi_i^{(\ell)}$ to (RW), the quantities

$$\begin{aligned} H_\ell[\Psi_1] &= \partial_r^{\ell+2}\Psi_1^{(\ell)} + \sum_{j=0}^{\ell+1} c_1^j \cdot \partial_r^j \Psi_1^{(\ell)} \\ H_\ell[\Psi_2] &= \partial_r^{\ell}\Psi_2^{(\ell)} + \sum_{j=0}^{\ell-1} c_2^j \cdot \partial_r^j \Psi_2^{(\ell)} \end{aligned} \tag{2}$$

are conserved along the null generators of $\mathcal{H}^+$.

## Decay along $\mathcal{H}^+$

- $\left|\partial_r^k \Psi_i^{(\ell)}\right| \lesssim \tau^{-1}, \quad \forall\ k \leq (\ell-2)-(-1)^i$
- $\left|\partial_r^{\ell-1-(-1)^i} \Psi_i^{(\ell)}\right| \lesssim \tau^{-\frac{3}{4}}$
- $\left|\partial_r^{\ell-(-1)^i} \Psi_i^{(\ell)}\right| \lesssim \tau^{-\frac{1}{4}}$

## Blow-up along $\mathcal{H}^+$

$\partial_r^{k+s_i}\Psi_i^{(\ell)}(\tau) = c_{i,k} H_\ell[\Psi_i] \cdot \tau^k + \mathcal{O}\left(\tau^{k-\frac{1}{4}}\right)$

for all $k \in \mathbb{N}$, where $s_i = \ell + 1 - (-1)^i$ ($c_{i,k}$ has alternating sign depending on $k \in \mathbb{N}$)

# Estimates for the Regge-Wheeler tensorial system

- Using the estimates obtained for $\Psi_i^{(\ell)},\ \ell \geq i,\ i \in \{1, 2\}$ we can also control $\phi_\ell, \psi_\ell$, since we can write

$$\phi_\ell = c_1 \cdot \Psi_1^{(\ell)} + c_2 \cdot \Psi_2^{(\ell)}, \qquad \psi_\ell = d_1 \cdot \Psi_1^{(\ell)} + d_2 \cdot \Psi_2^{(\ell)}$$

  In particular, they both inherit the behavior of the dominant term $\Psi_2^{(\ell)}$ asymptotically on $\mathcal{H}^+$.

- Using standard elliptic identities we obtain
  - Decay: $\|\boldsymbol{p}\|_{S^2_{\tau,M}} \lesssim_M \tau^{-\frac{3}{4}}, \qquad \left\|\nabla_{\partial_r}\boldsymbol{p}\right\|_{S^2_{\tau,M}} \lesssim_M \tau^{-\frac{1}{4}}$
  - Non-Decay: $\left\|\nabla^2_{\partial_r}\boldsymbol{p}\right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} c\,\|H_2[\Psi_2]\|_{S^2_{\tau,M}}$
  - Blow-up: $\left\|\nabla^k_{\partial_r}\boldsymbol{p}\right\|_{S^2_{\tau,M}} \gtrsim_M \|H_2[\Psi_2]\|_{S^2_{\tau,M}} \cdot \tau^{k-2}, \quad k \geq 3.$

  where $\|\xi\|^2_{S^2_{v,r}} := \int_{S^2_{v,r}} r^2 \sin\theta d\theta d\phi\,|\xi|^2$, and $|\xi|^2 := \xi^{A_1\dots A_n} \cdot \xi_{A_1\dots A_n}$.
  Similar estimates hold for $\boldsymbol{q}^F$.

# Estimates for the Teukolsky system

Recall the transformation relating Teukolsky solutions to Regge-Wheeler ones,

e.g. $$\boldsymbol{q}^F = r\nabla\!\!\!/_{\partial_r}\left(c_1(r)\cdot\mathfrak{f}\right)$$

Thus, taking $k$ many $\nabla\!\!\!/_{\partial_r}$ derivatives of the above we obtain

$$\nabla\!\!\!/_{\partial_r}^{k+1}\mathfrak{f} = \tilde{c}_1(r)\cdot\nabla\!\!\!/_{\partial_r}^{k}\boldsymbol{q}^F + \mathcal{L}^{k-1}[\boldsymbol{q}^F] + k_1(r)\cdot\mathfrak{f}$$

After obtaining decay estimates for $\mathfrak{f}$ using the transport equation (e.g.), we obtain

- Decay: $$\left\|\nabla\!\!\!/_{\partial_r}^{k}\mathfrak{f}\right\|_{S^2_{\tau,M}} \lesssim_M \frac{1}{\tau^{\left(\frac{4-k}{4}\right)^k}}, \qquad 0\le k\le 2.$$
- Non-Decay: $$\left\|\nabla\!\!\!/_{\partial_r}^{3}\mathfrak{f}\right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} c\left\|H_{\ell=2}[\Psi_2]\right\|_{S^2_{\tau,M}}$$
- Blow-up: $$\left\|\nabla\!\!\!/_{\partial_r}^{k}\mathfrak{f}\right\|_{S^2_{\tau,M}} \gtrsim_{k,M} \left\|H_{\ell=2}[\Psi_2]\right\|_{S^2_{\tau,M}}\cdot\tau^{k-3}, \qquad \forall\ k\ge 4.$$

Identical estimates hold for the gauge invariant quantity $\mathfrak{b}$ as well.

# Estimates for the Teukolsky system

Finally, in order to obtain estimates for the extreme curvature component $\alpha$, we use an induced identity relating all 3 gauge invariant quantities.

$$\nabla\!\!\!/_{\partial_r}(r\alpha) = c_1(r)\cdot \nabla\!\!\!/\widehat{\otimes}\mathfrak{b} + c_2(r)\cdot\mathfrak{f}$$

- $\left\|\nabla\!\!\!/^k_{\partial_r}\alpha\right\|_{L^\infty(S^2_{\tau,M})} \lesssim_M \dfrac{1}{\tau}, \qquad 0\le k\le 2.$

  $\left\|\nabla\!\!\!/^{k+2}_{\partial_r}\alpha\right\|_{L^\infty(S^2_{\tau,M})} \lesssim_M \dfrac{1}{\tau^{\left(\frac{4-k}{4}\right)^k}}, \qquad 1\le k\le 2.$

- $\left\|\nabla\!\!\!/^5_{\partial_r}\alpha\right\|_{S^2_{\tau,M}} \xrightarrow{\tau\to\infty} \left(\tilde{c}_2\left\|H_2[\Psi_2]\right\|^2_{S^2_{\tau,M}} + \tilde{c}_3\left\|H_3[\Psi_2]\right\|^2_{S^2_{\tau,M}}\right)^{\frac{1}{2}}$

- $\left\|\nabla\!\!\!/^k_{\partial_r}\alpha\right\|_{S^2_{\tau,M}} \gtrsim_{k,M} \tau^{k-5}, \qquad \forall\ k\ge 6.$

# Comparison

**On the horizon instability of an extreme Reissner-Nordström black hole**

James Lucietti[a*], Keiju Murata[b,c†], Harvey S. Reall[b‡] and Norihiro Tanahashi[d§]

April 24, 2013

- Moncrief's formalism. ( Newman–Penrose (NP) formalism)

| | $l = 1$ odd | $l > 1$ odd | $l = 1$ even | $l > 1$ even |
|---|---|---|---|---|
| $\psi$ | $P_f$ | $P_\pm$ | $H$ | $R_\pm$ |
| $W\vert_{r=M}$ | 6 | $l(l+1)+1\pm(2l+1)$ | 6 | $l(l+1)+1\pm(2l+1)$ |
| $p$ | 2 | $l\pm 1$ | 2 | $l\pm 1$ |

Table 1: Conserved quantities $H_p[\psi]$ for Moncrief's perturbations.

- For $\ell = 2$ odd parity perturbations the conserved quantities are

$$H_1[P_-] = \left(\partial_r^2 P_- + \frac{2}{M}\partial_r P_-\right)_{r=M}$$

$$H_3[P_+] = \left(\partial_r^4 P_+ + \ldots\right)_{r=M}$$

- $P_+ \sim \Psi_1^{(\ell)}$ and $P_- \sim \Psi_2^{(\ell)}$

Thank You
For Your Attention