

Hyperbolic Energy and Gluings of Initial Data

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based on joint work with P. T. Chruściel and E. Delay
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1. Theorem and Motivation
2. Solutions of Interest
3. Energy and Positive Mass Theorems
4. Asymptotically Locally Hyperbolic Manifolds with Negative Mass and Higher Genus

1. Theorem and Motivation

Asymptotically Locally Hyperbolic Manifolds with negative Mass

Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

$$R(g) = -6$$

*with connected conformal boundary at infinity, arbitrarily high genus and **negative total mass**.*

Asymptotically Locally Hyperbolic Manifolds with negative Mass

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- ▶ metric approaches a hyperbolic metric at large distances
- ▶ no interior boundary, only conformal boundary at infinity
- ▶ time-symmetric vacuum initial data with negative cosmological constant

Why is this interesting?

- ▶ provides better understanding of positivity of energy for asymptotically locally hyperbolic spaces
- ▶ hyperbolic space appears as constant time slice of Anti-de Sitter
- ▶ statement about initial data sets for asymptotically locally AdS spacetimes
- ▶ potential uses of bounds in AdS/CFT

2. Solutions of Interest

Birmingham-Kottler metrics

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

$$g_{3+1} = -V^2(r)dt^2 + \frac{1}{V^2(r)}dr^2 + r^2 h_k, \quad V^2(r) = r^2 + k - \frac{2m_c}{r}$$

with

$$k = \{-1, 0, 1\}$$

h_k is t - and r -independent Einstein metric on 2d compact, orientable manifold

$$R(h_k) = 2k$$

- ▶ $m_c \neq 0$ are singular unless $V(r_0) = 0$ for some $r_0 > 0$, if $V(r)$ has positive zero \rightarrow black hole solutions
- ▶ $m_c = 0$, $k = 1$ global AdS spacetime, $t = \text{const.}$ global hyperbolic space
- ▶ $m_c = 0$: locally AdS spacetime, $t = \text{const.}$ locally hyperbolic space

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- ▶ in space-dimension 3 asymptotically BK equivalent to asymptotically locally hyperbolic
- ▶ mass proportional to m_c , measured relativ to $b = g(m_c = 0)$

Horowitz-Myers metrics

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

$$g_{3+1} = V^2(r)d\theta^2 + \frac{1}{V^2(r)}dr^2 + r^2(-dt^2 + d\psi^2),$$

$$V^2(r) = r^2 - \frac{2m_c}{r}$$

- ▶ for $m_c > 0$, function $V(r)$ vanishes at $r = r_0 \rightarrow$ axis of rotation
- ▶ period of θ has to be chosen such that no conical singularity at $r = r_0$
 - ▶ period of θ depends on $m_c \rightarrow$ conformal infinity changes if m_c changes
- ▶ mass is proportional to $-m_c$ if measured with respect to BK with $m_c = 0$

Horowitz-Myers conjecture

$$g_{3+1} = V^2(r)d\theta^2 + \frac{1}{V^2(r)}dr^2 + r^2(-dt^2 + d\psi^2),$$

$$V^2(r) = r^2 - \frac{2m_c}{r}$$

- ▶ conjecture (1998): Horowitz-Myers metric minimizes the energy if you prescribe conformal structure at infinity

3. Energy and Positive Mass Theorems

How is the mass defined?

- ▶ often done using space-time methods
- ▶ also possible using initial data (M, g, K) : if g approaches Birmingham-Kottler metric with $m_c = 0$ ¹

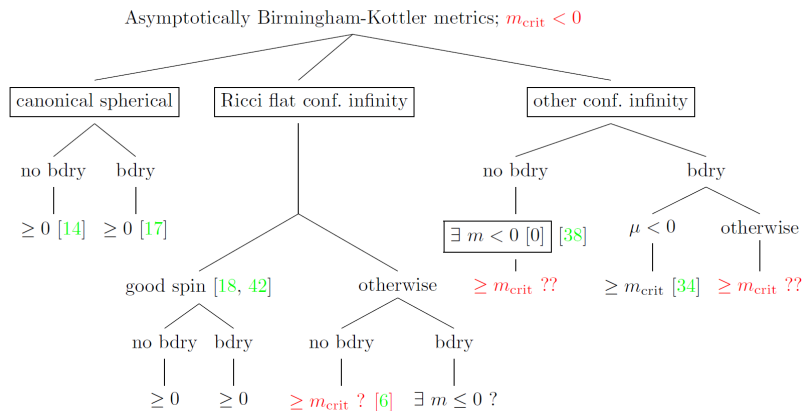
$$E = -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{r=R} D^j(V) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i$$

- ▶ R^i_j Ricci tensor of g
- ▶ g is the spatial part of the metric
- ▶ background enters through function V

$$V = \sqrt{r^2 + k}, \quad k \in \{-1, 0, 1\}$$

¹ $dS_i = \sqrt{\det g} \partial_i] dr \wedge d\theta \wedge d\psi$

Energy for asymptotically locally hyperbolic



- ▶ negative mass solutions in toroidal case: Horowitz-Myers

4. Asymptotically Locally Hyperbolic Manifolds with Negative Mass and Higher Genus

Idea: glue together two HM metrics at infinity

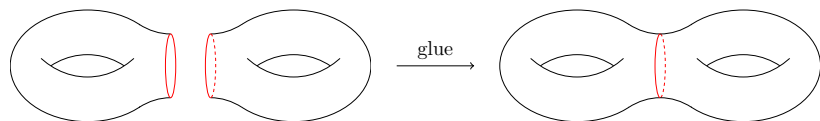
Theorem (Isenberg, Lee & Stavrov 2010,
)

Given two asymptotically hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity.

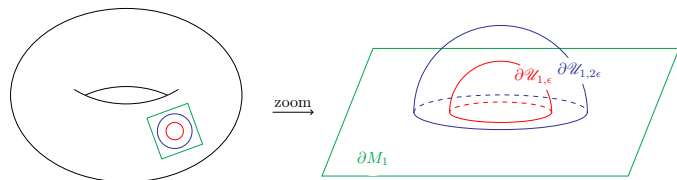
Idea: glue together two HM metrics at infinity

Theorem (Isenberg, Lee & Stavrov 2010, Chruściel, Delay 2015)

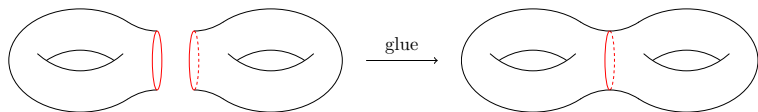
*Given two asymptotically hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be **localized** by a Carlotto-Schoen type hyperbolic gluing.*



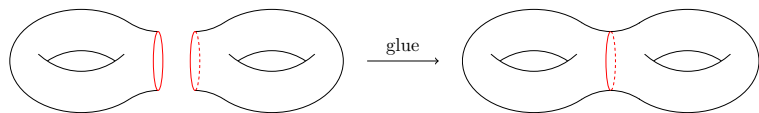
Idea: glue together two HM metrics at infinity



- ▶ metric is exactly hyperbolic inside red half-ball
- ▶ outside blue half-ball metric is exactly what it was before (e.g. Horowitz-Myers in our case)
- ▶ hyperbolic metric can be smoothly extended



Idea: glue together two HM metrics at infinity



- ▶ initial mass is defined with respect to a toroidal Birmingham-Kottler metric, final mass is defined with respect to a genus-2 Birmingham-Kottler metric

How does the mass change?

- ▶ initial toroidal background:

$$b = \frac{dr^2}{r^2} + r^2 \underbrace{(d\theta^2 + d\varphi^2)}_{h_0}$$

- ▶ final genus-2 background:

$$\bar{b} = \frac{d\bar{r}^2}{\bar{r}^2 - 1} + \bar{r}^2 \underbrace{(d\bar{\theta}^2 + \sinh^2(\bar{\theta})d\bar{\varphi}^2)}_{h_{-1}}$$

- ▶ on each half $h_{-1} = e^\omega h_0$
- ▶ initial mass is defined with respect to b , final mass is defined with respect to \bar{b}
- ▶ one can show that $\bar{r} = e^{-\frac{\omega}{2}} r + \text{subleading}$

How does the mass change?

- ▶ a few slides before we had

$$E_{generic} = -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{r=R} D^j(V) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i$$

with

$$V = \sqrt{r^2 + k}, \quad k \in \{-1, 0, 1\}$$

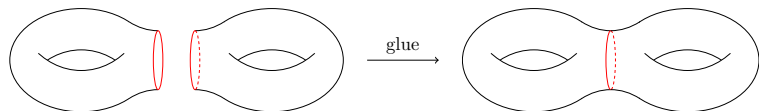
- ▶ mass of the initial torus

$$E = -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{r=R} D^j(r) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i$$

- ▶ mass of each half of the glued manifold

$$\begin{aligned} E &= -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{\bar{r}=R} D^j(\sqrt{\bar{r}^2 - 1}) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i \\ &= -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{r=R} D^j(e^{-\omega/2} r) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i \end{aligned}$$

Gluing tori and controlling the mass



- ▶ mass on each half of the manifold depends upon the gluing region
- ▶ sign of the final mass a priori unclear as both the metric and the conformal factor ω depend on the gluing region ϵ

$$E = -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{\{r=R\} \times T^2 \setminus D(p, \epsilon)} D^j(e^{-\omega/2} r) \left(R^i_j - \frac{R}{3} \delta^i_j \right) dS_i$$

- ▶ ϵ small needed

Taking the limit $\epsilon \rightarrow 0$

Theorem (P. T. Chruściel, E. Delay, RW)

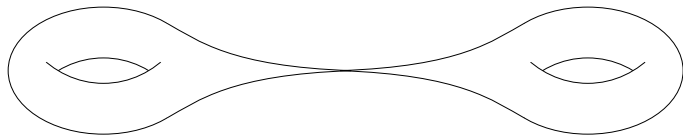
Upon gluing two Horowitz-Myers metrics with coordinate mass m_c , $e^\omega \rightarrow e^{\omega_0}$ of a punctured torus as $\epsilon \rightarrow 0$ with

$$E = -\frac{1}{8\pi} m_c \int_{T^2} e^{-\omega_0/2} d\mu_{h_0}$$

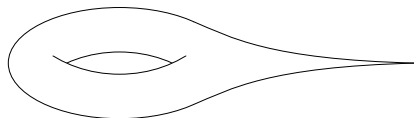
- ▶ it follows that if ϵ is chosen small enough, gluing of two Horowitz-Myers metrics gives genus-2 metrics with negative mass

What happens to geometry in limit $\epsilon \rightarrow 0$

- ▶ necks become thinner and longer as $\epsilon \rightarrow 0$

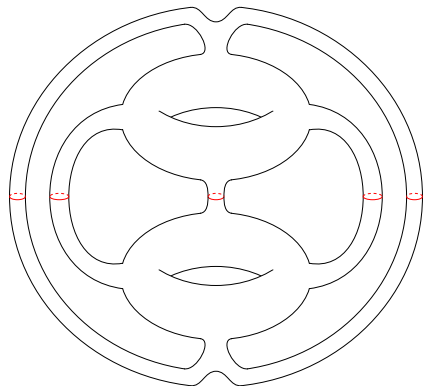


- ▶ as $\epsilon \rightarrow 0$ tori separate: two punctured tori



Generalizations

- ▶ construction can be iterated



Conclusion

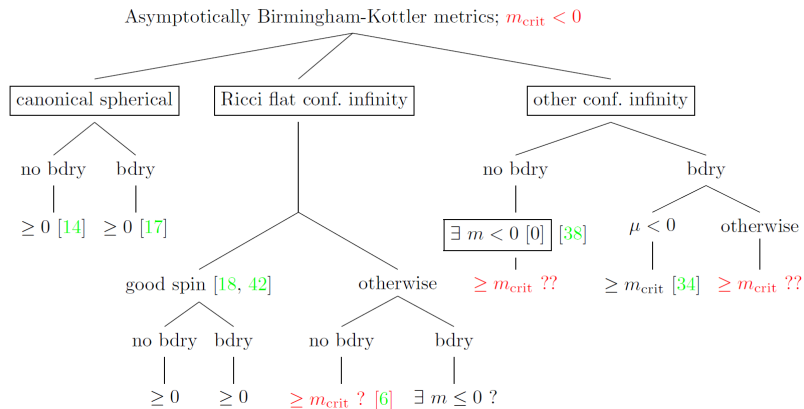
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Conclusion



Thank You!

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