Dynamics of Loop Quantum Cosmology

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GR23, July 5, 2022

Based on the joint work with You Ding, Gaoping Long, Jinsong Yang, Xiangdong Zhang, el.

Outline

Ideas of LQG

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Hence the theory of quantum gravity is a quantum theory of spacetime geometry with background independence.

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  Hence the theory of quantum gravity is a quantum theory of spacetime geometry with background independence.
- ⋆ The choice of the algebra of field functions to be quantized: The holonomies of the gravitational connection and the electric flux:

$$h_e(A) = \mathcal{P}\exp\int_e A_a, \qquad E(S,f) := \int_S \epsilon_{abc} E^a_i f^i$$

- There is a unique gauge and diffeomorphism invariant cyclic representation of the holonomy-flux $*-$algebra, given by the Ashtekar-Lewandowski measure μ_{AL} [Lewandowski, Okolow, Sahlmann, Thiemann, 2005].

Geometric Operators

- Area operator
  [Rovelli and Smolin, 1995; Ashtekar and Lewandowski, 1997]
  Given a closed 2-surface or a surface S with boundary, its area can be well defined as a self-adjoint operator $\hat{A}_S$ on $\mathcal{H}_{kin}$:

$$\hat{A}_S \psi_\alpha = 4\pi\gamma\ell_p^2 \sum_{v\in V(\alpha\cap S)} \sqrt{(\hat{J}^{(S,v)}_{i(u)} - \hat{J}^{(S,v)}_{i(d)})(\hat{J}^{(S,v)}_{j(u)} - \hat{J}^{(S,v)}_{j(d)})\delta^{ij}}\ \psi_\alpha .$$

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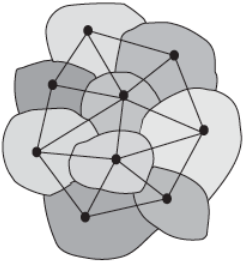
The typical eigenvalues of $\hat{A}_S$ are given by finite sums,

$$a_S = 8\pi\gamma\ell_p^2 \sum_I \sqrt{j_I(j_I+1)},$$

where j_I are arbitrary half-integers labeling the irreducible representations on the edges of spin networks.

Physical Meaning of Spin Networks

$|\Gamma, j_l, v_n\rangle$



Symmetric Reduction

- The idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried out in the symmetry-reduced models, known as Loop Quantum Cosmology.
- One freezes all but a finite number of degrees of freedom by imposing symmetries.
  The simplified framework provides a simple arena to test ideas and constructions.

Symmetric Reduction

- The idea that one should view holonomies rather than connections as basic variables for the quantization of gravity is successfully carried out in the symmetry-reduced models, known as Loop Quantum Cosmology.
- One freezes all but a finite number of degrees of freedom by imposing symmetries.
  The simplified framework provides a simple arena to test ideas and constructions.
- Symmetries: homogeneity and (or) isotropy.
- Example: Spatially flat FRW universe
  - Spatial 3-manifold: $\mathbb{R}^3$
  - Isometry: Euclidean group

The Kinematical Setting of LQC

- One has to introduce an elementary cell $\mathcal{V}$ and restricts all integrations to this cell.
- Fix a fiducial flat metric ${}^{o}q_{ab}$ and denote by V_o the volume of $\mathcal{V}$ in this geometry.
  The gravitational phase space variables —the connections and the density weighted triads — can be expressed as
  $A_a^i = c\, V_o^{-(1/3)}\ {}^{o}\omega_a^i$ and $E_i^a = p\, V_o^{-(2/3)} \sqrt{{}^{o}q}\ {}^{o}e_i^a$,
  where $({}^{o}\omega_a^i, {}^{o}e_i^a)$ are a set of orthonormal co-triads and triads compatible with ${}^{o}q_{ab}$ and adapted to $\mathcal{V}$.

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  where $({}^{o}\omega^i_a, {}^{o}e^a_i)$ are a set of orthonormal co-triads and triads compatible with ${}^{o}q_{ab}$ and adapted to $\mathcal{V}$.
- To pass to the quantum theory, one constructs a kinematical Hilbert space $\mathcal{H}^{\rm grav}_{\rm kin} = L^2(\mathbb{R}_{\rm Bohr}, {\rm d}\mu_{\rm Bohr})$, where $\mathbb{R}_{\rm Bohr}$ is the Bohr compactification of the real line and ${\rm d}\mu_{\rm Bohr}$ is the Haar measure on it.

The Improved Scheme

- It is convenient to introduce new conjugate variables by a canonical transformation [Ashtekar, Pawlowski, Singh, 2006]:

$$b := \frac{\sqrt{\Delta}}{2}\frac{c}{\sqrt{|p|}}, \qquad \nu := \frac{4}{3\sqrt{\Delta}} sgn(p)|p|^{\frac{3}{2}},$$

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- In the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$, eigenstates of $\hat{\nu}$, which are labeled by real numbers v, constitute an orthonormal basis as: $\langle v_1|v_2\rangle = \delta_{v_1,v_2}$.
- The fundamental operators act on $|v\rangle$ as: $\hat{\nu}\,|v\rangle = (8\pi\gamma l_p^2/3)v|v\rangle$ and $\widehat{e^{ib}}\,|v\rangle = |v+1\rangle$.

Dynamical Setting

- The gravitational Hamiltonian constraint in the connection formulation of GR is given by

$$H_{\rm grav}(1) = \int_{\mathcal{V}} d^3x \frac{E^{aj}E^{bk}}{2\kappa\sqrt{\det(q)}}\left[\epsilon_{ijk}F^i_{ab} - 2(1+\gamma^2)K^j_{[a}K^k_{b]}\right]$$
$$\equiv H^E(1) - 2(1+\gamma^2)\,T(1),$$

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  where F_{ab} is the curvature of A^i_a, and K^i_a is the extrinsic curvature of the spatial hypersurface.

- In the model of spatially homogenous and isotropic cosmology, the Euclidean term $H^E(1)$ is proportional to the Lorentzian term $T(1)$, so that

$$H^E_{\rm grav}(1) = -\int_{\mathcal{V}} d^3x \frac{\epsilon^{ijk}E^a_jE^b_k}{2\kappa\gamma^2\sqrt{\det(q)}}\epsilon_{ilm}A^l_aA^m_b.$$

APS Dynamics for LQC

- Starting from the classical Hamiltonian $H^E_{\mathrm{grav}}(1)$, the first physically reasonable gravitational Hamiltonian operator was given by [Ashtekar, Pawlowski, Singh, 2006]:

$$\hat{C}_{\mathrm{grav}}\,|v\rangle = f_+(v)|v+4\rangle + f_o(v)|v\rangle + f_-(v)|v-4\rangle,$$

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  which contributes a second order difference equation for the quantum dynamics.
- Phenomenological issues of LQC have been studied based on this dynamics, e.g,
  - the probability of inflation [Ashtekar, Sloan, 2011],
  - imprints on CMB [Agullo, Ashtekar, Nelson, 2012; Ashtekar, Gupta, 2016].

Quantum Bounce

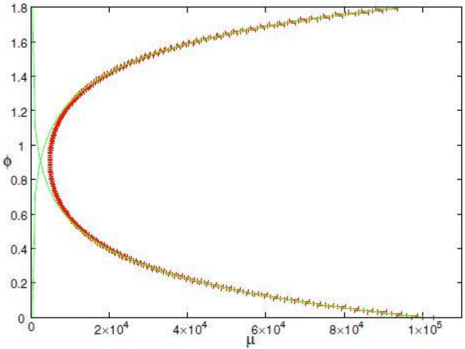

FIG. 2 (color online). The expectation values of Dirac observables $\hat{\mu}|_\phi$ are plotted (in multiples of μ_o), together with their dispersions. They exhibit a quantum bounce which joins the contracting and expanding classical trajectories marked by fainter lines.

Alternative Hamiltonian operator for LQC

- Hamiltonian constraint operators can be well defined in certain Hlibert spaces in LQG, e.g., in $\mathcal{H}_{kin}$ or $\mathcal{H}^G$ [Thiemann 1996], in $\mathcal{H}_{\rm vtx}$ [Lewandowski, Sahlmann, 2014], in $\mathcal{H}_{\rm np4}$ [Yang and YM, 2015].
- By Thiemann's trick for full LQG, the Lorentzian term can be written as

$$T(1) = -\frac{2}{\kappa^4\gamma^3}\int_{\mathcal{V}} d^3x\,\tilde{\epsilon}^{abc}\mathrm{Tr}\left(\{A_a, K\}\{A_b, K\}\{A_c, V\}\right)$$

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- LQC gravitational Hamiltonian operator with Lorentzian and Euclidean terms [Yang, Ding, YM, 2009]:

$$\begin{aligned}\hat{H}^F_{\rm grav}|v\rangle = \quad & F'_+(v)|v+8\rangle + f'_+(v)|v+4\rangle + \left(F'_o(v) + f'_o(v)\right)|v\rangle \\ & + f'_-(v)|v-4\rangle + F'_-(v)|v-8\rangle.\end{aligned}$$

Alternative Effective Hamiltonian

- The action of the alternative Hamiltonian constraint operator $\hat{H}^F_{\rm grav}$ contributes a fourth order difference equation for the quantum dynamics with step of different size comparing to that of the original APS Hamiltonian operator.

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- The effective Hamiltonian of $\hat{H}_F = \hat{H}^F_{\mathrm{grav}} + \hat{H}_\phi$ with the relevant quantum corrections can be derived as [Yang, Ding, YM, 2009]

$$\mathcal{H}^F_{\mathrm{eff}} = -\frac{3\beta}{\gamma^2\kappa\Delta}|v|sin^2(2b)\left(1-\left(1+\gamma^2\right)\sin^2(2b)\right) + \beta|v|\rho.$$

where $\beta = \dfrac{\kappa\hbar\gamma\sqrt{\Delta}}{4}$, and ρ is the density of the matter field.

Modified Dynamical Equations

- The modified Friedmann and Raychaudhuri equations can be derived from the effective Hamiltonian $\mathcal{H}^F_{\rm eff}$ as

$$H^2 = \frac{1}{\gamma^2\Delta}\sin^2(b)(1-\sin^2(b))(1-2(1+\gamma^2)\sin^2 b)^2,$$

$$\frac{\ddot{a}}{a} = H^2 + \frac{1}{\gamma\sqrt{\Delta}}\dot{b}(1-2\sin^2 b-2(1+\gamma^2)\sin^2 b(3-4\sin^2 b)),$$

where H denotes the Hubble parameter.

Alternative Effective Hamiltonian

- The Hamiltonian $\mathcal{H}^F_{\text{eff}}$ can be reproduced by a suitable semiclassical analysis of Thiemann's Hamiltonian in full LQG, but in the μ_0 scheme [Dapor, Liegener, 2018; Han, Liu, 2020].
- Different from that of APS dynamics, the alternative dynamics from $\hat{H}_F$ leads to an asymmetric quantum bounce. [Assanioussi, Dapor, Liegener, Pawlowski, 2018]
- An emergent quasi deSitter space is present on the other side of the bounce, but the emergent cosmological constant is very large with a reasonable gravitational constant [Li, Singh, Wang, 2018].

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- An emergent quasi deSitter space is present on the other side of the bounce, but the emergent cosmological constant is very large with a reasonable gravitational constant [Li, Singh, Wang, 2018].
- The phenomenological implications of $\mathcal{H}^F_{\text{eff}}$ is being studied, e.g.,
    - The inflationary phase can naturally take place with a very high probability [Li, Singh, Wang, 2018].
    - The power spectra of the cosmological perturbations is studied, and in certain approach its difference from that of APS model is relevant to the current observations [Li's talk].
    - It might restrict a cyclic evolution of the universe [Singh's talk].

Alternative Dynamics for LQC

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- The equivalent effective $f(R)$ theory corresponding to $\mathcal{H}^F_{\text{eff}}$ is found [Ribeiro, Vernieri, Lobo, 2021].
- The idea of treating the Lorentzian term separately can also be used in the scheme of using the Chern-Simons action to regularize the Hamiltonian constraint and obtain a new Hamiltonian operator for LQC [Yang, Zhang, YM, 2020].
- The idea has also been used in loop quantum Brans-Dicke cosmology to obtain alternative dynamics for the theory. [Song, Zhang, YM, 2020]

New Perspective

- The general expression of the gravitational Hamiltonian constraint for LQG can be written as [Zhang, Long, YM, 2021]

$$H_g = \lambda H^E - (1+\lambda\gamma^2)H^L + (-1+\lambda)\int_\Sigma \sqrt{q}R,$$

where λ is an arbitrary real number to represent the freedom of choices.

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where λ is an arbitrary real number to represent the freedom of choices.

- For the spatially flat model, $R = 0$, the Hamiltonian constraint reduces to

$$\begin{aligned} H_g &= \lambda H^E - (1+\lambda\gamma^2)H^L \\ &= \frac{1}{2\kappa}\int_\Sigma \left[\lambda F^j_{ab} - (\lambda\gamma^2+1)\varepsilon_{jmn}K^m_a K^n_b\right]\frac{\varepsilon^{jkl}E^a_k E^b_l}{\sqrt{q}} \end{aligned}$$

Quantum Dynamics

- By coupling to certain matter field, the effective Hamiltonian constraint of the LQC model can be obtained as

$$\mathcal{H}^{\lambda}_{\text{eff}} = -\frac{3\beta}{\gamma^2\kappa\Delta}|v|\sin^2 b\left(1-(1+\lambda\gamma^2)\sin^2(b)\right)+\beta|v|\rho.$$

- The quantum bounce takes place at the point of $\sin^2(b) = \dfrac{1}{2(1+\lambda\gamma^2)}$ with $\rho = \dfrac{\rho_c}{4(1+\lambda\gamma^2)}$.

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- The quantum bounce takes place at the point of $\sin^2(b) = \frac{1}{2(1+\lambda\gamma^2)}$ with $\rho = \frac{\rho_c}{4(1+\lambda\gamma^2)}$.
- The asymptotic behaviours of the effective dynamics in the large v region can be derived for

$$b \to 0 \quad or \quad b \to \arcsin\left(\frac{1}{\sqrt{(1+\lambda\gamma^2)}}\right) = b_0$$

Asymptotic Friedman Equations

- Asympototicaly the effective Friedman equations read

$$
\begin{aligned}
H^2 &= \frac{\kappa}{3}\rho, \qquad (b \to 0) \\
H^2 &= \frac{1-5\lambda\gamma^2}{1+\lambda\gamma^2}\frac{\kappa\rho}{3} + \frac{\Lambda}{3}, \qquad (b \to b_0)
\end{aligned}
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with $\Lambda = \dfrac{3\lambda}{(1+\lambda\gamma^2)^2\Delta}$.

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with $\Lambda = \dfrac{3\lambda}{(1+\lambda\gamma^2)^2\Delta}$.

- For a chosen $\lambda > 0$, the effective Hamiltonian of the spatially flat FLRW leads to an asymptotic de-Sitter branch of the Universe connecting to the standard Friedmann branch by the quantum bounce.

Emergent Cosmological Constant

- At large volume limit of the asymptotic de-Sitter branch, we obtain

$$\Lambda = \frac{3\lambda}{(1+\lambda\gamma^2)^2\Delta}, \qquad G_\lambda = \frac{1-5\lambda\gamma^2}{1+\lambda\gamma^2}G.$$

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- By choosing $\lambda = \frac{\Delta\Lambda_{ob}}{3}$, an effective cosmological constant $\Lambda_{ob} \sim 1.09 \times 10^{-52} m^{-2}$ consistent with the current observations can be obtained, and the effective gravitational constant G_λ also satisfies the experimental restrictions.
- Therefore, by this choice of regularization of the Hamiltonian constraint, the accelerating expansion of our universe (dark energy) could be attributed to the emergent effects of quantum gravity.

Summary

- ⋆ The idea of background independence is successful realized in the construction of LQG.
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- ⋆ The idea of background independence is successful realized in the construction of LQG.
- ⋆ The idea and technique of LQG have been successfully carried out in the symmetry-reduced models, known as LQC.
- The big bang singularity of classical GR can be resolved by a quantum bounce of LQC.
- Alternative Hamiltonian operators which are closely related to that of full LQG have been proposed for LQC.
- The phenomenological issues of the alternative effective dynamics of LQC are being studied with possible observational effects.
- In a new perspective, the so-called dark energy could be attributed to an emergent effect of LQC.

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