# Analytic studies of power spectra of cosmological perturbations in loop quantum cosmology

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   - Uniform Asymptotic Approximation(UAA) for two turning points

2. Applicaton
   - Pöschl-Teller (PT) potential
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   - Compare Exact solution and UAA solution

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# The Liouville Transformation

In general, by properly choosing the variable $y$ and $\mu_k(y)$, the second order differential equation can be written as

$$\frac{d^2 u_k(y)}{dy^2} = f(y)\mu_k(y),$$

and $f(y)$ can take the form

$$f(y) = \lambda^2 g(y) + q(y).$$

# The Liouville Transformation

For the equation

$$\frac{d^2\mu_k(y)}{dy^2} = [\lambda^2 g(y) + q(y)]\mu_k(y), \tag{1}$$

we can have the Liouville transformation with two variables $U(\zeta)$ and $\zeta(y)$ (with its inverse $y = y(\zeta)$) satisfying

$$U(\zeta) = \dot{y}^{-\frac{1}{2}}\mu_k(y), \quad \dot{y} = \frac{dy}{d\zeta}.$$

# The Liouville Transformation

In terms of $U(\zeta)$ and $\zeta$, the equation (1) can be written as

$$\frac{d^2U(\zeta)}{d\zeta^2} = [\lambda^2 \dot{y}^2 g + \psi(\zeta)]U(\zeta), \tag{2}$$

where,

$$\begin{aligned}\psi(\zeta) &\equiv \dot{y}^2 q - \dot{y}^{\frac{1}{2}} \frac{d^2}{d\zeta^2}\left(\dot{y}^{-\frac{1}{2}}\right) \\ &= \dot{y}^2 q - \dot{y}^{\frac{3}{2}} \frac{d^2}{dy^2}\left(\dot{y}^{\frac{1}{2}}\right) \equiv \psi(y).\end{aligned}$$

In the equation

$$\frac{d^2U(\zeta)}{d\zeta^2} = [\lambda^2\dot{y}^2 g + \psi(\zeta)]U(\zeta),$$

considering $\psi(\zeta) = 0$ as the first-order approximation, we can choose $\dot{y}^2 g$ such that:

- the resulting equation can be solved explicitly,
- the (first-order) approximation can be as close as possible to the exact solution.

The choice of $\dot{y}^2 g$ sensitively depends on the behavior of the function $g(y)$ and $q(y)$ near their poles and turning points (zeros), as well as the natures of theses points.

# UAA for two turning points

We are focusing on the cases with two turning points. For this case, $g(y)$ can always be written as

$$g(y) = p(y)(y - y_1)(y - y_2),$$

where p(y) is regular in the interval of interest.

We can divide all cases in three different classes:

- $y_1$ and $y_2$ are two real and distict roots of $g(y) = 0$,
- $y_1 = y_2$, a double root of $g(y) = 0$,
- $y_1$ and $y_2$ are two complex roots of $g(y) = 0$, and $y_1 = y_2^*$.

There are three conditions are assumed to be satisfied when applying the UAA methods:

- When far away from any turning points:
$|\frac{q(y)}{g(y)}| \ll 1$,
- When near any of these two points and the two turning points are far way from each other ($|y_1 - y_2| \gg 1$):
$|\frac{q(y)(y-y_i)}{g(y)}| \ll 1$ ,
- When near these points and the two turning points are close to each other ($|y_1 - y_2| \approx 1$):
$|\frac{q(y)(y-y_1)(y-y_2)}{g(y)}| \ll 1$ .

# UAA for two turning points

In particular, we choose

$$\dot{y}^2 g = \zeta_0^2 - \zeta^2, \quad (3)$$

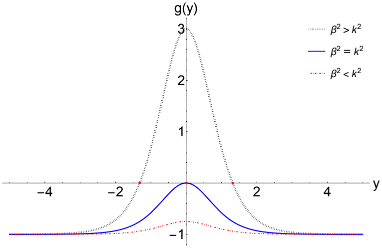
where $\zeta(y)$ is an increasing function with the choices $\zeta(y_1) = -\zeta_0, \zeta(y_2) = \zeta_0$.

And it can be shown that

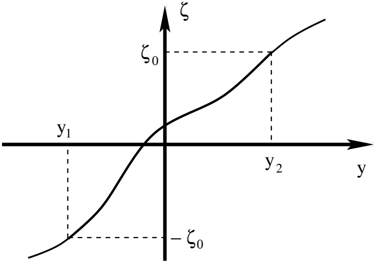
$$\zeta_0^2 = \pm\frac{2}{\pi}\left|\int_{y_1}^{y_2}\sqrt{g(y)}dy\right|,$$

where "$+$" and "$-$" corresponds to the cases with two real turning points and two complex conjugate turning points respectively.

# UAA for two turning points




(a) $g(y)$




(b) $\zeta(y)$

# UAA for two turning points

With the choice of (3), we find that (2) reduces to

$$\frac{d^2U}{d\zeta^2} = [\lambda^2(\zeta_0^2 - \zeta^2) + \psi(\zeta)]U.$$

In the first-order approximation, $\psi(\zeta)$ is neglectable, we can have the approximate solutions in the form of the parabolic cylinder functions $W(s,t)$:

$$U(\zeta) = \alpha_1 \left[ W(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta) + \epsilon_1 \right] + \beta_1 \left[ W(\frac{1}{2}\lambda\zeta_0^2, -\sqrt{2\lambda}\zeta) + \epsilon_2 \right], \quad (4)$$

# UAA for two turning points

from which, we have the approximate solutions to the original equation (1):

$$\begin{aligned}\mu_k(y) =& \alpha_1 \left(\frac{\zeta^2-\zeta_0^2}{-g(y)}\right)^{\frac{1}{4}} \left[W(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta) + \epsilon_1\right] \\ &+ \beta_1 \left(\frac{\zeta^2-\zeta_0^2}{-g(y)}\right)^{\frac{1}{4}} \left[W(\frac{1}{2}\lambda\zeta_0^2, -\sqrt{2\lambda}\zeta) + \epsilon_2\right].\end{aligned} \tag{5}$$

# UAA for two turning points

The errors $\epsilon$ have some bounded conditions

$$\frac{|\epsilon_1|}{M(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta)}, \frac{|\frac{\partial \epsilon_1}{\partial \zeta}|}{\sqrt{2}N(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta))} \leq \frac{1}{\lambda E(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta))}\left(e^{\lambda V_\zeta} - 1\right),$$

$$\frac{|\epsilon_2|}{M(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta)}, \frac{|\frac{\partial \epsilon_2}{\partial \zeta}|}{\sqrt{2}N(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta))} \leq \frac{E(\frac{1}{2}\lambda\zeta_0^2, \sqrt{2\lambda}\zeta))}{\lambda}\left(e^{\lambda V_\zeta} - 1\right).$$

where $V_\zeta$ is the associated error control function for the approximate solutions,

$$V_\zeta = \int^\zeta \frac{|\psi(\zeta)|}{\sqrt{|\dot{y}^2 g|}} d\zeta.$$

# Pöschl-Teller (PT) potential

To test the analytical approximate solutions, we consider equation (1) with a Pöschl-Teller (PT) potential

$$(\lambda^2 g + q) = -(k^2 - \frac{\beta_0^2}{\cosh^2 y}),$$

where $k$ is the comoving wavenumber and $\beta_0$ is a real and positive constant.

# Pöschl-Teller (PT) potential

In this case, the equation (1) becomes

$$\frac{d^2\mu_k(y)}{dy^2} + (k^2 - \frac{\beta_0^2}{\cosh^2 y})\mu_k(y) = 0.$$

And with the variable transformations:

$$x = \frac{1}{1+e^{-2y}}, \quad \mathcal{Y} = [x(1-x)]^{\frac{ik}{2}}\mu_k,$$

we find that (1) with PT potential reads

$$x(1-x)\frac{d^2\mathcal{Y}(x)}{dx^2}+(a_3-(a_1+a_2+1)x)\frac{d\mathcal{Y}(x)}{dx}-a_1a_2\mathcal{Y}(x)=0, \quad (6)$$

where

$$a_1=\frac{1}{2}(1+\sqrt{1-4\beta_0^2})-ik,$$
$$a_2=\frac{1}{2}(1-\sqrt{1-4\beta_0^2})-ik,$$
$$a_3=1-ik.$$

The equation (6) is the standard hypergeometric equation, and has the general solution

$$\mu_k^{(E)}(y) = a_1\left(\frac{x}{1-x}\right)^{\frac{ik}{2}} \times {}_2F_1(a_1 - a_3 + 1, a_2 - a_3 + 1, 2 - a_3, x)$$
$$+ b_1[x(1-x)]^{-\frac{ik}{2}} \times {}_2F_1(a_1, a_2, a_3, x).$$

# UAA solutions

First, we choose the $g(y)$ and $q(y)$ as

$$g(y) = \frac{\beta^2}{\cosh^2 y} - k^2 \quad q(y) = \frac{q_0^2}{\cosh^2 y},$$

then it turns out that

$$|\frac{q(y)}{g(y)}| \sim q_0^2 e^{-2|y|},$$
$$|\frac{q(y)(y-y_i)}{g(y)}| \sim \frac{q_0^2}{y+y_j},$$
$$|\frac{q(y)(y-y_1)(y-y_2)}{g(y)}| \sim q_0^2.$$

So as long as $q_0$ chosen to be small, we can use UAA method to get the approximate solution.

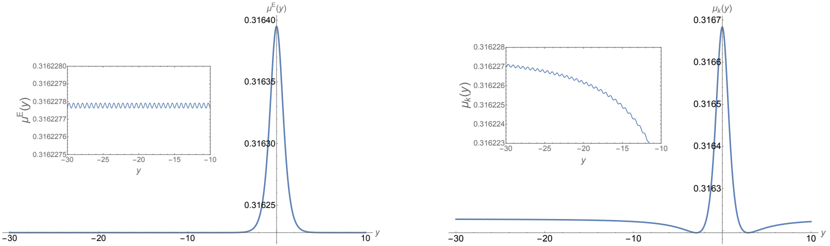
# UAA solutions

Applying the UAA method, we can have the approximate solutions:

$$\mu_k(y) = \alpha_k \left( \frac{\zeta^2 - \zeta_0^2}{-g(y)} \right)^{\frac{1}{4}} W(\frac{\zeta_0^2}{2}, \sqrt{2}\zeta) + \beta_k \left( \frac{\zeta^2 - \zeta_0^2}{-g(y)} \right)^{\frac{1}{4}} W(\frac{\zeta_0^2}{2}, -\sqrt{2}\zeta).$$

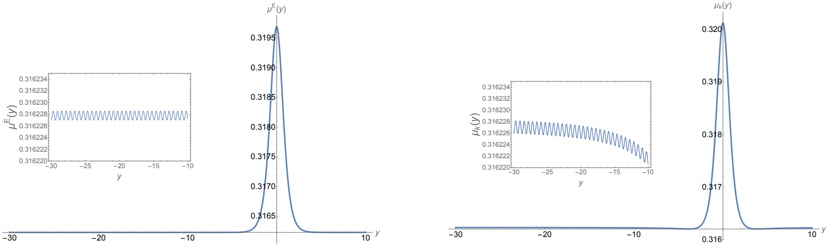
# Compare Exact solution and UAA solution

Now we can compare the exact and approximate solutions. First we plot both solutions in the range of interest.




(c) Exact (d) UAA

Figure 1: $k = 5.0, \beta = 0.1$ and $q_0^2 = 1/24$



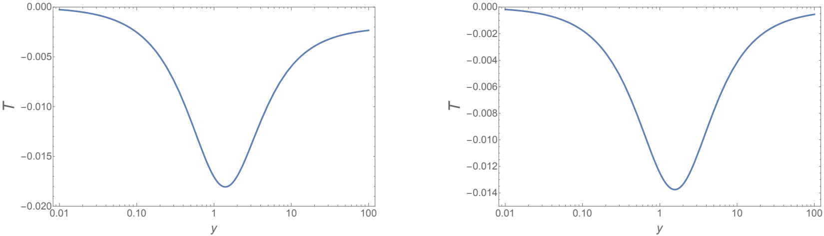

(a) Exact (b) UAA

Figure 2: $k = 5.0, \beta = 1.0$ and $q_0^2 = 1/24$

The error control function $V_\zeta$ is given by

$$V = -\int^y \left(\frac{q}{g} - \frac{5}{16}\frac{g'^2}{g^3} + \frac{1}{4}\frac{g''}{g^2}\right)\sqrt{-g}dy' + \frac{\zeta(\zeta^2 - 6\zeta_0^2)}{12\zeta_0^2(\zeta^2 - \zeta_0^2)^{3/2}},$$

and the plots are:



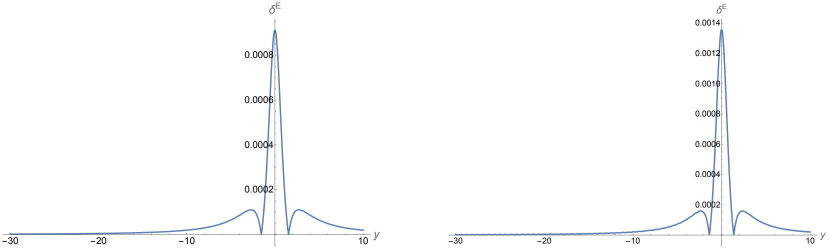

(a) $k \gg \beta$ (b) $k > \beta$

Figure 3: Error control function $V_\zeta$

We define the relative difference $\delta^E$ between approximate solution and exact solution as

$$\delta^E = \left| \frac{|\mu_k(y)| - |\mu_k^{(E)}(y)|}{|\mu_k^{(E)}(y)|} \right|,$$

then we have,




(a) $k \gg \beta, \quad \delta < 0.09\%$ (b) $k > \beta, \quad \delta < 0.14\%$

Figure 4: $\delta^E(y)$

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# Discussion and Summary

- The UAA method also applys to the cases $k \sim \beta$(currently working) and $k \ll \beta$ (single turning point[1]) with modifications;
- The UAA method provides a useful way to solve second-order differential equations by approximate solutions, associated with some particular bounded controlled errors, especially when the equation is hard to solve analytically;
- Applying to PT potential case indicates that UAA approximation has small enough errors($< 0.15\%$), which is accurate enough for studying the cosmology observation data($1 \sim 2\%$).

# References

[1]F.W.J. Olver, *The asymptotic solution of linear differential equations of the second order in a domain containing one transition point*, Philos. Trans. Roy. Soc. London, A **249** (1956) 65.
[2]F. W. J. Olver, 6 - THE LIOUVILLE-GREEN APPROXIMATION, in *Introduction to Asymptotics and Special Functions*, edited by F. W. J. Olver (Academic Press, 1974), pp. 190–228.
[3]T. Zhu, A. Wang, G. Cleaver, K. Kirsten, and Q. Sheng,*Pre-inflationary universe in loop quantum cosmology*, Phys. Rev. D **96** 083520 (2017).
[4]E. Berti, V. Cardoso, and A. O. Starinets, *Quasinormal Modes of Black Holes and Black Branes*, Class. Quantum Grav. 26, 163001 (2009).

Thank you!