Finiteness of the spin foam vertex amplitude with timelike polyhedra and the full amplitude

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Lorentzian spin foam amplitudes ($\Lambda = 0$)

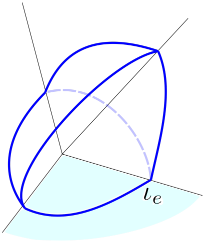
1. General triangulation consists of simplices with tetrahedra of both **euclidean** and **lorentzian** signature.
2. Standard EPRL-FK model implements only **euclidean** ones. Conrady-Hnybida (CH) extension for **lorentzian**. In this talk I will restrict only to a special case when **all triangles are spacelike**.
3. In this model:

$SU(2) \subset SL(2,\mathbb{C})$ timelike normal	⟷	$SU(1,1) \subset SL(2,\mathbb{C})$ spacelike normal
$D_j \subset \mathcal{H}_{j,\rho}$ Sub-representations	⟷	Discrete series $D_j^{\pm}$ future/past normals to the triangles

4. Different kinds of tetrahedra can coexist in a spin foam. Extension to cellular complexes also possible (and considered in this talk).

Recently extensively studied: asymptotics, **finiteness**.

Vertex amplitude EPRL-FK



1. Vertex spin network Γ_v: intersection of a small sphere around the vertex with the complex. Edges $e \supset v$ induce nodes, faces $f \supset v$ induce links (edges of the graph).
2. The nodes are decorated by invariants $\iota_e \in \bigotimes_{f \supset e} D_{j_f}$ for $e \supset v$.
3. We embed ι_e into tensor product of $\mathcal{H}_{j_f,\rho_f}$ (labelled by $f \supset e$),

The vertex amplitude is the contraction according to the graph

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})^{|N_v|-1}} \prod_{e \supset v} dg_e \,\mathrm{contr} \bigotimes_{e \supset v} g_e \triangleright \iota_e$$

where one integration is ommited ($g_{e_0} = \mathbb{I}$).

Is it finite (integrable)?

Spin foam amplitude EPRL-FK

Results in EPRL-FK model [Baez-Barret '01, WK '10]:

1. The vertex amplitude is finite (integrable) for every 3-edge connected graph. Graph is n-edge connected if we cannot separate it by cutting $n-1$ edges/links.
2. Group theoretic consideration shows that graph of lower connectivity cannot give integrable amplitude, thus the result is sharp.

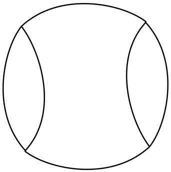
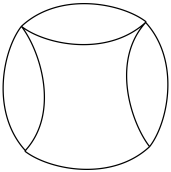


Figure: 2-edge connected graph.



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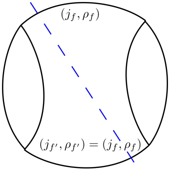

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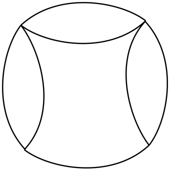

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Spin foam amplitude CH extension

Conrady-Hynbida extension:

1. **What is the space of invariants in $\bigotimes_i D^+_{j_i^+} \otimes \bigotimes_i D^-_{j_i^-}$?** They do not longer belong to the tensor product. Invariants are functionals on a dense subspace, so called **algebraic tensor product**

$$\text{alg. span} \bigotimes_i v_i^+ \otimes \bigotimes_i v_i^-, \quad v_i^\pm \in D^\pm_{j_i^\pm} \tag{1}$$

   Other choices of subspace are also possible.

2. **Is the space of invariants finite dimensional?** No.
   Semiclassically, there are arbitrary large tetrahedra with triangles of arbitrary small areas ($-++$ signature).

3. **Is the vertex amplitude finite?**

Space of invariants (description)

Decomposition $(D^+_{j_1} \otimes D^+_{j_2} = \bigoplus_{j=j_1+j_2}^{\infty} D^+_j)$

$$\bigotimes_i D^{\pm}_{j_i^{\pm}} = \bigoplus_k n_k^{\pm} D_k^{\pm}, \quad \kappa_k^{\pm n} \colon \bigotimes_i D^{\pm}_{j_i^{\pm}} \to D_k^{\pm} \tag{2}$$

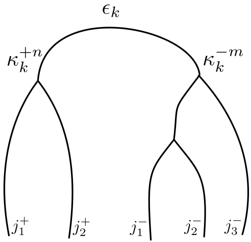
Maps $\kappa_k^{\pm n}$ can be obtained by branching. Duality map

$$\epsilon_k \colon D_k^+ \otimes D_k^- \to \mathbb{C} \tag{3}$$

The basis in the space of invariants (labelled by k and multiplicities $n.m$)

$$\iota_{k,n,m} := \epsilon_k(\kappa_k^{+n} \otimes \kappa_k^{-m}) \tag{4}$$

- Space infinite dimensional unless at most one future or past normals.
- The **shadow** eigenvalue k: Semiclassical interpretation k^2 as maximal projection of tetrahedron on the plane.
- For fixed k finite dimensional space.



Invariants via group averaging

The basis in the space of invariants

$$\iota_{k,n,m} := \epsilon_k(\kappa_k^{+n} \otimes \kappa_k^{-m}) \tag{5}$$

1. Importantly, it can be obtained via group averaging

$$\iota_{k,n,m} = c \int_{\mathrm{SU}(1,1)} dg \; g \triangleright (\langle k|\kappa_k^{+n} \otimes \langle -k|\kappa_k^{-m}) \tag{6}$$

2. We will write it as

$$c\Big[\langle k|\kappa_k^{+n} \otimes \langle -k|\kappa_k^{-m}\Big] \tag{7}$$

   The **representant** (vector which is group averaged) belongs to an algebraic tensor product of representations.

3. The basis is orthogonal with respect to group averaging scalar product.

$$\langle [I], [I'] \rangle_{av} := [I]^\dagger (I') \tag{8}$$

Finiteness of the vertex amplitude

Vertex amplitude

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})^{|N_v|-1}} \prod_{e \supset v} dg_e \,\mathrm{contr} \bigotimes_{e \supset v} g_e \triangleright I_e, \quad \iota_e = [I_e] \tag{9}$$

1. The result does not depend on the choice of representants I_e of ι_e.
2. Uniform treatment of both EPRL and CH models.
3. The integrand is well-defined if I_e belong to the algebraic tensor product of $\mathrm{Irrep}\,\mathrm{SL}(2,\mathbb{C})$.
4. One $\mathrm{SL}(2,\mathbb{C})$ integration is omitted (result does not depend on this choice).

Is it finite?

Finiteness of the vertex amplitude

We introduce a useful class of functions:

1. Irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C})$ can be realized as homogenous functions on $\mathbf{z} \in \mathbb{C}^2 \setminus \{0\}$.

$$L^\infty = \{f \in \mathcal{H}_{j,\rho} \colon \sup_{|\mathbf{z}|=1} |f| < \infty\} \tag{9}$$

2. Functions which are in L^∞ also under duality transformation $\hat{\beta} \colon \overline{\mathcal{H}_{j,\rho}} \to \mathcal{H}_{j,\rho}$,

$$L^\infty \cap \hat{\beta}(L^\infty) \tag{10}$$

3. Algebraic tensor product

$$\hat{L}_e^\infty = \text{ alg. span} \bigotimes_{f \supset e} L^\infty \cap \hat{\beta}(L^\infty) \tag{11}$$

This will be the class of functions for the representants of invariants.

Finiteness of the vertex amplitude

Theorem [Han, WK, Liu]

If every $I_e \in \hat{L}_e^\infty$ and the vertex graph is 3-edge connected then the vertex amplitude is integrable (finite).

1. The assumption about invariants is true in the case of both EPRL/FK as well as Conrady-Hnybida model with spacelike faces.
2. Proof by majorization through Barrett-Crane amplitude

Summary and outlook

Summary

- The vertex amplitude is finite for spin foam models with arbitrary normals but all triangles spacelike.
- The spin foam amplitude is finite for fixed spins if we impose additional cut-off by shadow.

Outlook:

- What with timelike triangles (signature $+-$)?
- Is the shadow cut-off really necessary?

Thank you!