# Scalar curvature operator for loop quantum gravity on a cubical graph

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# Scalar curvature in loop quantum gravity

The object of interest: Ricci scalar integrated over the spatial manifold

$$\int_\Sigma d^3x \sqrt{q}\, {}^{(3)}R$$

Relevant to loop quantum gravity

- As a geometrical observable
- As an alternative to the Lorentzian part of the Hamiltonian constraint

$$C = \frac{1}{\beta^2} \frac{\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab}}{\sqrt{|\det E|}} + (1+\beta^2)\sqrt{|\det E|}\, {}^{(3)}R$$

Previously: The "Regge" curvature operator (Alesci, Assanioussi, Lewandowski 2014)

$$\int d^3x \sqrt{q}\, {}^{(3)}R \simeq \sum_{\text{hinges}} (\text{hinge length}) \times (\text{deficit angle})$$

Refers classically to an auxiliary manifold of singular geometry (curvature concentrated on one-dimensional hinges) instead of the actual physical manifold

# Ricci scalar as a function of the Ashtekar variables

The starting point of our construction: Express the Ricci scalar as ${}^{(3)}R(A, E)$

$$q^{ab} = \frac{E_i^a E_i^b}{|\det E|} \qquad {}^{(3)}R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc}$$

$$\begin{aligned}
|\det E|\, {}^{(3)}R = & -2E_i^a \mathcal{D}_{(a}\mathcal{D}_{b)} E_i^b + 2Q^{ab} E_c^i \mathcal{D}_a \mathcal{D}_b E_i^c \\
& - (\mathcal{D}_a E_i^a)(\mathcal{D}_b E_i^b) - \frac{1}{2}(\mathcal{D}_a E_i^b)(\mathcal{D}_b E_i^a) \\
& + \frac{5}{2} Q^{ab} (\mathcal{D}_a E_i^c)(\mathcal{D}_b E_c^i) - \frac{1}{2} Q^{ab} Q_{cd} (\mathcal{D}_a E_i^c)(\mathcal{D}_b E_i^d) \\
& + 2\mathcal{A}^{ab}{}_a \mathcal{B}_{cb}{}^c + 2\mathcal{A}^{ab}{}_b \mathcal{B}_{ca}{}^c + \mathcal{A}^{ab}{}_c \mathcal{B}_{ba}{}^c \\
& + \frac{1}{2} Q_{ab} \mathcal{A}^{ca}{}_d \mathcal{A}^{db}{}_c - Q^{ab} \mathcal{B}_{ca}{}^c \mathcal{B}_{db}{}^d \\
& + 2\left(Q^{ab}\mathcal{B}_{ca}{}^c - \mathcal{A}^{ab}{}_a - \mathcal{A}^{ba}{}_a\right) \frac{\partial_b |\det E|}{|\det E|} \\
& + \frac{3}{2} Q^{ab} \frac{\partial_a |\det E|}{|\det E|} \frac{\partial_b |\det E|}{|\det E|} - 2Q^{ab} \frac{\partial_a \partial_b |\det E|}{|\det E|}
\end{aligned}$$

$$Q^{ab} = E_i^a E_i^b$$
$$Q_{ab} = E_a^i E_b^i$$
$$\mathcal{A}^{ab}{}_c = E_i^a \mathcal{D}_c E_i^b$$
$$\mathcal{B}_{ab}{}^c = E_a^i \mathcal{D}_b E_i^c$$

Gauge covariant derivatives:

$$\mathcal{D}_a E^b = \partial_a E^b + \left[A_a, E^b\right]$$
$$\mathcal{D}_a \mathcal{D}_b E^c = \partial_a \left(\mathcal{D}_b E^c\right) + \left[A_a, \mathcal{D}_b E^c\right]$$

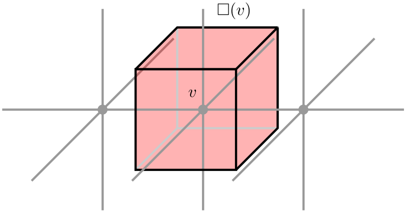
# Regularization on a cubical graph

Main assumption/simplification:

*Aim to construct the operator on the Hilbert space of a fixed cubical graph*

May seem like a severe restriction. However, several approaches in LQG make extensive use of states defined on cubical graphs:

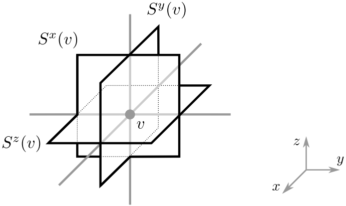
- Algebraic quantum gravity
- Quantum-reduced loop gravity
- Models of effective dynamics




The integrated Ricci scalar is regularized as a Riemann sum over the cubical partition:

$$\int d^3x\,\sqrt{q}\,{}^{(3)}R$$

$$\simeq \sum_{\square} \epsilon^3 \sqrt{|\det E|}(v)\,{}^{(3)}R(v)$$



# Regularization of gauge covariant derivatives

We use parallel transported flux variables (also known as gauge covariant fluxes) to regularize the gauge covariant derivatives of the triad.

$$\widetilde{E}(S, x_0) = \int_S d^2\sigma \, n_a(\sigma) h_{x(\sigma)\to x_0} E^a\big(x(\sigma)\big) h^{-1}_{x(\sigma)\to x_0}$$

The definition involves choosing a family of paths which connect every point $x(\sigma)$ on the surface to a fixed point $x_0$ (on the surface or outside of it).

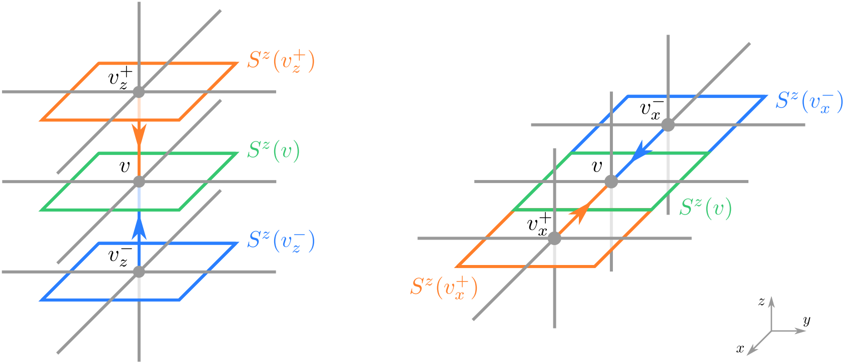
Covariant derivatives of the triad are approximated as finite differences of neighboring parallel transported fluxes (all transported to the same node $v$).

First derivatives: $$f'(x) \simeq \frac{f(x+\epsilon) - f(x-\epsilon)}{2\epsilon}$$

Second derivatives: $$f''(x) \simeq \frac{f(x+\epsilon) - 2f(x) + f(x-\epsilon)}{\epsilon^2}$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \simeq \frac{f(x+\epsilon, y+\epsilon) - f(x+\epsilon, y-\epsilon) - f(x-\epsilon, y+\epsilon) + f(x-\epsilon, y-\epsilon)}{4\epsilon^2}$$

# Regularization of gauge covariant derivatives

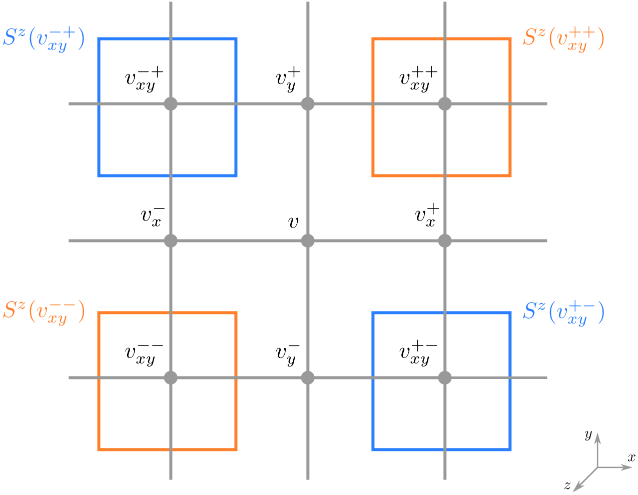



$$\Delta_a E\big(S^b, v\big) \equiv \frac{\widetilde{E}\big(S^b(v_a^+), v\big) - \widetilde{E}\big(S^b(v_a^-), v\big)}{2}$$

$$\Delta_{aa} E\big(S^b, v\big) \equiv \widetilde{E}\big(S^b(v_a^+), v\big) - 2\widetilde{E}\big(S^b(v), v\big) + \widetilde{E}\big(S^b(v_a^-), v\big)$$

$$\Delta_a E\big(S^b, v\big) \simeq \epsilon^3 \mathcal{D}_a E^b(v) \qquad \Delta_{aa} E\big(S^b, v\big) \simeq \epsilon^4 \mathcal{D}_a^2 E^b(v)$$

# Mixed second derivatives

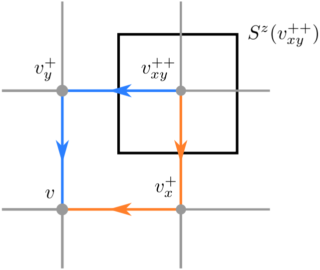



$$\frac{\widetilde{E}\big(S^c(v_{ab}^{++}), v\big) - \widetilde{E}\big(S^c(v_{ab}^{+-}), v\big) - \widetilde{E}\big(S^c(v_{ab}^{-+}), v\big) + \widetilde{E}\big(S^c(v_{ab}^{--}), v\big)}{4}$$

# Symmetric part of the mixed second derivative

Two equally good paths are available for parallel transport to the central node. Hence introduce

$$\widetilde{E}\big(S^z(v_{xy}^{++}),v\big)_{\text{sym.}} \equiv \frac{1}{2}\Big(\widetilde{E}\big(S^z(v_{xy}^{++}),v\big)_{v_{xy}^{++}\to v_x^{+}\to v} + \widetilde{E}\big(S^z(v_{xy}^{++}),v\big)_{v_{xy}^{++}\to v_y^{+}\to v}\Big)$$




Then we define the discretized derivative as

$$\Delta_{ab}E\big(S^c,v\big) \equiv \frac{1}{4}\Big(\widetilde{E}\big(S^c(v_{ab}^{++},v\big)_{\text{sym.}} - \widetilde{E}\big(S^c(v_{ab}^{+-},v\big)_{\text{sym.}} - \widetilde{E}\big(S^c(v_{ab}^{-+},v\big)_{\text{sym.}} + \widetilde{E}\big(S^c(v_{ab}^{--},v\big)_{\text{sym.}}\Big)$$

This approximates the symmetric part of the mixed second derivative at $v$:

$$\Delta_{ab}E\big(S^c,v\big) \simeq \epsilon^4\mathcal{D}_{(a}\mathcal{D}_{b)}E^c(v)$$

# Quantization

After regularization on the cubical lattice, we have

$$\sqrt{q}\,{}^{(3)}\!R = \mathcal{R}\big(E^a, \mathcal{D}_a E^b, \mathcal{D}_a \mathcal{D}_b E^c, \sqrt{|\det E|}, \partial_a \sqrt{|\det E|}\big)$$

$$\int d^3x\,\sqrt{q}\,{}^{(3)}\!R \simeq \sum_{\square} \mathcal{R}\Big(\widetilde{E}(S^a), \Delta_a E(S^b), \Delta_{ab} E(S^c), V(\square), \Delta_a V(\square)\Big)$$

Now every factor can be promoted into an operator in LQG.

Negative powers of the volume are quantized using the regularized inverse volume operator:

$$\frac{1}{V(\square)} \quad \longrightarrow \quad \widehat{\mathcal{V}_v^{-1}} \equiv \lim_{\delta\to 0} \frac{\hat{V}_v}{\hat{V}_v^2 + \delta^2}$$

(Previous uses in LQG: Length operator, Regge curvature operator, Warsaw Hamiltonian)

The result is an operator of the form

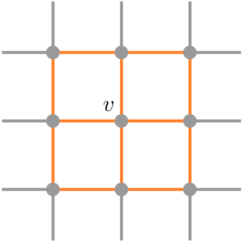
$$\int \widehat{d^3x\,\sqrt{q}\,{}^{(3)}\!R} \;=\; \sum_{v\in\Gamma_{\rm cubical}} \hat{\mathcal{R}}_v$$

on the Hilbert space of the chosen cubical graph.

# Properties of the curvature operator

- Gauge invariant (under internal $SU(2)$ transformations)
- Adjoint operator is densely defined on the Hilbert space of the fixed graph. Hence it is possible to construct a symmetric factor ordering (e.g. for the physical Hamiltonian in deparametrized models)
- Action of the operator

$\hat{\mathcal{R}}_v :$




  The degree of complexity is roughly similar to the Euclidean part of Thiemann's Hamiltonian in the usual graph-preserving regularization

- The operator acts by coupling holonomies of spin 1. There is no regularization ambiguity related to the spin

# Conclusions

We have defined a new operator representing the three-dimensional scalar curvature in loop quantum gravity.

- The operator is restricted to the Hilbert space of a fixed cubical graph
- Classical starting point: Ricci scalar expressed directly as a function of the densitized triad and its gauge covariant derivatives
- The covariant derivatives are quantized by using parallel transported flux variables to discretize them on the cubical lattice provided by the graph

Open questions/topics for future work:

- Generalize the construction to other kinds of graphs (e.g. four-valent nodes)
- Extension to the entire kinematical Hilbert space of LQG? Modified regularization of derivatives needed to obtain a symmetric operator
- Curvature operator for quantum-reduced loop gravity (work in progress)
- Semiclassical properties?
- Physical applications?

Thank you for your attention!