Entropy of black holes with arbitrary shapes in loop quantum gravity

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Content

Motivation

- The computation of black hole (BH) entropy from basic principles is an essential test for any candidate theory of quantum gravity. In the loop quantum gravity (LQG) framework, then entropies of the type I and type II isolated horizons (IHs) have been studied. How to calculate the entropy of type III IHs is still an important open issue. [Beetle and Engle, 2010, Perez and Pranzetti, 2011]
- BF theory explanation of IH entropy does not require the symmetries of IH [Wang et al., 2014]. It is possible to generalize this approach to all three types IH. But we should distinguish different horizons with the same area.

A short review on the BH entropy in LQG

- Boundary degrees of freedom:
  - Chern-Simons theory approach:

    gauge groups: $U(1) \to SU(2)$ [Engle et al., 2010]

    symmetries: type I IH $\to$ type II IH [Ashtekar et al., 2005]

    distorsion [Perez and Pranzetti, 2011]
  - BF theory approach: SO(1,1), SU(2) [Pranzetti and Sahlmann, 2015]

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- state counting method: generating function method [Sahlmann, 2008, Agullo et al., 2008, Agullo et al., 2010, Barbero G. and Villasenor, 2008]

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- Ensembles: microcanonical and canonical ensembles [Ghosh and Perez, 2011]

BF theory approach of type I IH

- BF theory: $S[A,B] = \int_\Sigma B \wedge F(A)$. $SO(1,1)$ BF theory is under consideration. Since one has $SO(1,1) \simeq \mathbb{R}$, the connection field A is a real-valued 1-form, and the B field is also a real-valued 1-form. Considering the spacetime admitting IH as internal boundary, the presymplectic form of the boundary on the covariant phase space is

$$\Omega(\delta_1, \delta_2) = 2 \oint_H \delta_{[2} B \wedge \delta_{1]} A. \tag{1}$$

- The internal degree of freedom tetrad of IHs in GR case is $SO(1,1)$ after fixing the horizons and their foliations. The presymplectic form is reduced to

$$\Omega(\delta_1, \delta_2) = 2 \oint_H \delta_{[2} B \wedge \delta_{1]} \bar{\omega}^{01}. \tag{2}$$

where $dB = \frac{\Sigma_{01}}{8\pi G}$. According to the presymplectic form, the internal degree of freedom tetrad of IH in GR can be regarded as a punctured BF theory[Wang et al., 2014].

- Let us assume that the graph Γ underlying a spin network state intersects H by n punctures denoted by $\mathcal{P} = \{p_i | i = 1, \cdots, n\}$. For every puncture p_i we associate a bounded neighborhood s_i which contains it and does not intersect any other. We denote the boundary of s_i by η_i. The physical degrees of freedom of our BF theory are encoded in the flux functions

$$f_i = \int_{s_i} dB = \oint_{\eta_i} B.$$

The quantum Hilbert space $\mathcal{H}_H^{\mathcal{P}}$ can be defined as the space of L^2 functions on $\mathbb{R}^n$. as configuration operators, $\hat{f}_i$ act on any wave function by multiplications. The common eigenstates of all these $\hat{f}_i$ are the Dirac distributions $(\{a_p\}, \mathcal{P}| \equiv (a_1, a_2, \cdots, a_n|$ characterized by n real numbers $\{a_i, i = 1, \cdots, n\}$.

$$(\{a_p\}, \mathcal{P}|\hat{f}_i = (\{a_p\}, \mathcal{P}|a_i. \qquad (3)$$

- flux-area operator [Fernando Barbero et al., 2009]

$$a^{flux}(\tilde{E}, r) = \frac{1}{2}\int_S \left|\tilde{E}^a_I r^I \epsilon_{abc} \mathrm{d}x^b \wedge \mathrm{d}x^c\right|. \quad (4)$$

the eigen value is

$$a^{flux}(m_1, \ldots, m_n) = 8\pi\gamma\ell_P^2 \sum_{i=1}^{n} |m_i|. \quad (5)$$

- Boundary conditions

$$(\mathrm{Id} \otimes \hat{f}_i(s_i) - \hat{\Sigma}_{01}(s_i) \otimes \mathrm{Id})(\Psi_H \otimes \Psi_b) = 0, \tag{6}$$

$$\int_H |\Sigma_{01}| = a_H. \tag{7}$$

give the constraints

$$\sum_{p \in \mathcal{P}} |m_p| = a, \quad m_p \in \mathbb{N}/2, a = a_H/(8\pi\gamma l_{Pl}^2) \tag{8}$$

- The number of quantum states is

$$\mathcal{N} = \sum_{n=0}^{n=2a-1} C_{2a-1}^n 2^{n+1} = 2 \times 3^{2a-1}, \tag{9}$$

where C_i^j are the binomial coefficients. The entropy is [Wang et al., 2014]

$$S = \ln \mathcal{N} = 2a \ln 3 + \ln \frac{2}{3} = \frac{\ln 3}{\pi\gamma} \frac{a_H}{4l_{Pl}^2} + \ln \frac{2}{3}. \tag{10}$$

The shape of IH

- Using the foliation given in Ref. [Ashtekar et al., 2002], an IH $(\Delta, [\ell], \tilde{q}_{ab}, \tilde{\omega}_a)$ can be foliated into 2-spheres with a unique geometric pair (q_{ab}, ω_a), where q_{ab} and ω_a are respectively the projections of $\tilde{q}_{ab}$ and $\tilde{\omega}_a$ of Δ to the 2-sphere H.
- For type II IHs, ω_a is corresponding to the angular momentum [Ashtekar et al., 2004], while the entropy is corresponding to the area.

The shape of IH

- Given a 2-sphere with a 2-metric q_{ab} and everywhere positive curvature, it can be globally immersed in the 3-dimensional Euclidean space $\mathbb{R}^3$, and q_{ab} can completely determine the extrinsic curvature (i.e., the extrinsic shape) of the 2-sphere immersed in $\mathbb{R}^3$ [Spivak, 1970].
- Any Riemann metric on a 2-sphere is conformal to a round metric. For a given horizon H of a BH with a spherical topology and any physical metric q, if the scalar curvature is positive almost everywhere, there exists a unique fiducial round metric $\mathring{q}$ on H, which is conformal to the physical metric [Ashtekar et al., 2022], i.e., $q_{ab} = \Omega^2 \mathring{q}_{ab}$, where the positive function Ω is called conformal factor.

The shape of IH

To regularize the area element, one can divide H into "small enough" patches $\{O^{(i)}\}$ such that each $O^{(i)}$ has the same area measured by $\mathring{q}$. For instance, the partition can be realized by triangulation. The total number K of the patches should satisfy $1 \ll K \ll \frac{a_H}{4\pi\gamma\ell_p^2}$, where a_H is the area of H measured by q. Let $4\pi\gamma\ell_p^2\, a^{(i)}$ be the physical area of $O^{(i)}$. By fixing once and for all a way to order the patches $\mathcal{O} \equiv \{O^{(1)}, O^{(2)}, \cdots, O^{(K)}\}$, we obtain a corresponding ordered area number sequence $\{a^{(1)}, a^{(2)}, \cdots, a^{(K)}\}$ with $\sum_{i=1}^{K} a^{(i)} = \frac{a_H}{4\pi\gamma\ell_p^2}$, which is called a "shape" of H with the total area a_H.

A way to assign the ordering of the patches is shown in Fig. 1. The intersections inside a patch $O^{(j)}$ contribute the sequence $v^{(j)} \equiv (v_1^{(j)}, v_2^{(j)}, \cdots, v_i^{(j)}, \cdots)$ to $O^{(j)}$.

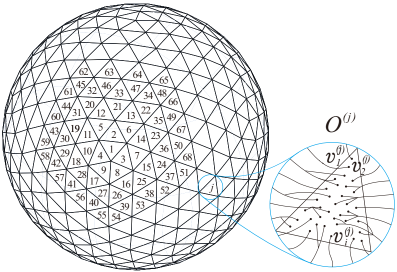

Figure 1: Ordering patches of H and their intersections with spin networks

Boundary condition

The SO(1,1) BF theory description does not require any symmetry on IH. This advantage enables us to calculate the statistic entropy of general IHs by the BF theory approach.

Then the dimension of the boundary Hilbert space $\mathcal{H}_H$ is given by the number of ordered sequence $(v_1^{(1)}, v_2^{(1)}, \cdots; v_1^{(2)}, v_2^{(2)}, \cdots; \cdots; v_1^{(K)}, v_2^{(K)}, \cdots)$ subject to the following piece-area constraints and projection constraint

$$\sum_i \left| v_i^{(j)} \right| = a^{(j)}, \quad \forall j, \tag{11}$$

$$\sum_{j=1}^{K} V^{(j)} = 0, \tag{12}$$

where a quantum number $V^{(j)} \equiv \sum_i v_i^{(j)}$ is defined for each patch $O^{(j)}$.

Generating function method

A generating function is a way of encoding an infinite sequence of numbers ($\mathcal{N}_n$) by treating them as the coefficients of a formal power series.

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- Example (partitions of an integer): The number of ordered partitions of n is the coefficient of the x^n term in the Taylor series expansion of the function $\frac{1}{1-\sum_{i=1}^{\infty} x^i}$, i.e.,

$$[x^n]\frac{1}{1-\sum_{i=1}^{\infty} x^i} = [x^n]\frac{1-x}{1-2x} = 2^{n-1}, \tag{13}$$

where we have used $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ for $|x| < 1$. The ordered partition of integer has the constraint

$$\sum_i v_i = n, \quad v_i \in \mathbb{N}^+. \tag{14}$$

Generating function method

- One-step function of each patch $O^{(j)}$ by

$$f(x_j, z) = \sum_{n_j=1}^{\infty} (z^{n_j} + z^{-n_j}) x_j^{n_j},$$

$z^{n_j} x_j^{n_j}$ means a intersection in patch $O^{(j)}$ contributes to both the area number and magnetic number by n_j, and $z^{-n_j} x_j^{n_j}$ contributes to the area number and magnetic number by n_j and $-n_j$ respectively.

- The generating function for $O^{(j)}$ could be expanded by power series of one-step function f or variables x_j and z, reads

$$\begin{aligned} G(x_j, z) &= 1 + \sum_{n=1}^{\infty} f^n(x_j, z) \\ &= 1 + \sum_{a^{(j)}=1}^{\infty} \sum_{V^{(j)}=-a^{(j)}}^{a^{(j)}} \mathcal{N}(a^{(j)}, V^{(j)}) x_j^{a^{(j)}} z^{V^{(j)}}, \end{aligned}$$

The microstate number for given $a^{(j)}$ and $V^{(j)}$ is

$$\mathcal{N}(a^{(j)}, V^{(j)}) = \begin{cases} A_i^n, & a^{(j)} = 2n\text{–}1, \ V^{(j)} = \pm(2i\text{–}1); \\ B_0^n, & a^{(j)} = 2n, \quad V^{(j)} = 0; \\ B_i^n, & a^{(j)} = 2n, \quad V^{(j)} = \pm 2i; \\ 0, & \text{others.} \end{cases}$$

with $n, \ i \in \mathbb{N}^+$, $i \leqslant n$ and

$$A_i^n = \sum_{j=0}^{n-i} \frac{(-3)^{j-1}(-6n+2j+3)2^{2n-2j-2}(2n-j-2)!}{j!(n-i-j)!(n+i-j-1)!},$$

$$B_0^n = \sum_{i=0}^{n} \frac{(-3)^{i-1}(-3n+i)2^{2n-2i}(2n-i-1)!}{i!((n-i)!)^2},$$

$$B_i^n = \sum_{j=0}^{n-i} \frac{(-3)^{j-1}(-3n+j)2^{2n-2j}(2n-j-1)!}{j!(n-i-j)!(n+i-j)!}.$$

Numerical calculation

For a given shape, one first compute the total microstate number for a possible quantum number sequence $\{V^{(j)}\}$ by neglecting the projection constraint as

$$\mathcal{N}\left(\{a^{(j)}\},\{V^{(j)}\}\right)=\prod_{j=1}^{K}\mathcal{N}(a^{(j)},V^{(j)}).$$

Then the total microstate number satisfying the projection constraint can be calculated by

$$\mathcal{N}\left(\{a^{(j)}\}\right)=\sum_{\{V^{(j)}\}}\left(P_{\{V^{(j)}\}}\cdot\mathcal{N}(\{a^{(j)}\},\{V^{(j)}\})\right),$$

where the projection constraint is realized by demanding

$$P_{\{V^{(j)}\}}=\begin{cases}0, & \sum_{j=1}^{K}V^{(j)}\neq 0,\\ 1, & \sum_{j=1}^{K}V^{(j)}=0.\end{cases}$$

The numerical computations of the entropy $S = \ln \mathcal{N}(\{a^{(j)}\})$ for small black holes indicate the formula:

$$S_0 = \frac{\ln 3}{\pi\gamma} \frac{a_H}{4\ell_P^2} - \frac{1}{2} \ln \frac{a_H}{4\gamma\ell_P^2} + K \ln \frac{2}{3}. \qquad (16)$$

This formula has the following convincing features.

- The coefficient $\frac{\ln 3}{\pi\gamma}$ of the leading order term in (16) matches the results in other approaches employing also the flux-area operator [Fernando Barbero et al., 2009, Wang et al., 2014].
- The coefficient $-\frac{1}{2}$ of the subleading logarithmic correction term matches the results of $U(1)$ Chern-Simons theory approaches [Fernando Barbero et al., 2009, Domagala and Lewandowski, 2004].
- The next order correction term containing K matches the result of $SO(1,1)$ BF theory approach for spherically symmetric IH with $K = 1$ [Wang et al., 2014].

Examples and error analysis

1) For spherically symmetric BH, $a^{(j)} = \tilde{a}$, given the partition number K, the absolute error $\Delta S = S - S_0$ decreases in inverse proportion as the area number $\tilde{a}$ increases as shown in Fig. 2.

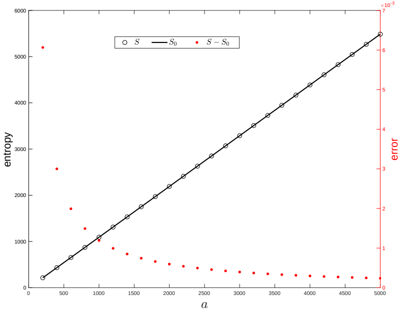

Figure 2: Numeric result of entropy with $k = 5$, $\tilde{a} \in [40, 1000]$ and the interval of $\tilde{a}$ is 40. Black dot is the actual entropy S, solid line is the expected entropy S_0, and the red dots are their difference.

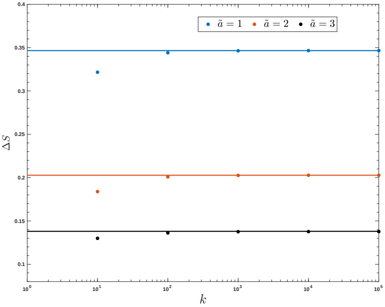

Figure 3: The tendency of absolute error (dots) with respect to k and the upper bounds (straight line) as $k \to \infty$. The color blue, orange and black represent piece area number $\tilde{a} = 1, 2, 3$ respectively.

For a given $\tilde{a}$, ΔS increases as K increases. Fig. 3 indicates that there is an upper bound for ΔS as K increases for a given $\tilde{a}$, and the upper bound of ΔS decrease as $\tilde{a}$ increases. Thus in the extreme case of $\tilde{a} = 1$, the upper bound takes the maximal value $\Delta S_{max} \approx \frac{1}{2} \ln 2$.

2) For different shapes

Table 1: Numerical results of S and ΔS for different sizes and shapes of small black holes

Shape	$\Delta S/10^{-3}$	Shape	$\Delta S/10^{-3}$
(80,80,80,80,80)	2.99987	(160,160,160,160,160)	1.49209
(40,80,160,80,40)	2.99987	(80,160,320,160,80)	1.49209
(30,70,200,70,30)	2.99987	(60,140,400,140,60)	1.49209
(196,196,4,2,2)	2.77401	(392,392,8,4,4)	1.48020
(396,1,1,1,1)	1.72774	(796,1,1,1,1)	0.86161
S_0	433.84949	S_0	872.94783

Summary

- The ordering area number sequence $\{a^{(1)}, a^{(2)}, \cdots, a^{(K)}\}$ was employed to characterize the shape of a BH. A delicate issue here is how to choose the partition number K for a given H. Thus one reasonable choice is to ask the number K to be proportional to the area a_H and fix its value by assigning a mesoscopic scale δ to $\sqrt{\frac{a_H}{K}}$. Then the entropy formula (16) becomes

$$S_0 = \Big(\frac{\ln 3}{\pi\gamma} + \frac{4\ell_P^2}{\delta^2}\ln\frac{2}{3}\Big)\frac{a_H}{4\ell_P^2} - \frac{1}{2}\ln\frac{a_H}{4\gamma\ell_P^2}. \tag{17}$$

  the very small number $\frac{4\ell_P^2}{\delta^2}\ln\frac{2}{3}$ can be regarded as a correction from quantum and semi-classical geometries. As a new quantum gravity effect, the latter might be fixed by other semi-classical consideration of LQG, for instance, the analysis in Ref. [Han and Zhang, 2016].

- For a given horizon area, the entropy decreases as a black hole deviates from the spherically symmetric. The shape gives higher order correction than the corrections showed in (17) to the entropy, which is worth further investigating.

Summary

- Using the ordered area number sequence $\{a^{(1)}, a^{(2)}, \cdots, a^{(K)}\}$ with $\sum_{i=1}^{K} a^{(i)} = \frac{a_H}{4\pi\gamma\ell_p^2}$ to quantify the shape of isolated horizon. For given area a_H, the amount of shapes is finite.
- The entropy S of any type isolated horizon satisfies $S_0 - \ln\frac{3}{\sqrt{2\pi}} \leqslant S < S_0 + \frac{1}{2}\ln 2$, where

$$S_0 = \frac{\ln 3}{\pi\gamma}\frac{a_H}{4\ell_P^2} - \frac{1}{2}\ln\frac{a_H}{4\gamma\ell_P^2} + K\ln\frac{2}{3},$$

  with K satisfies $1 \ll K \ll \frac{a_H}{4\pi\gamma\ell_p^2}$.
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Thank you for your attention!

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