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On quantum-induced corrections to the stress-energy tensor on Finsler manifold

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Physical Problem

Unification of GR with QM.

Quantization of GR but also Gravitization of QM

General Relativity

(roughed at some scale)

- Spacetime discretization,
- Measurement uncertainty,
- Noncommutative relations,
- Generalized Riemann manifold

Quantum Mechanics

(smoothed at some scale)

- Gravitational field impacts,
- Generalized noncommutitavity,
- Relativity principle,
- Isotropy & Lorentz covariance

A minimum measurable length sets limits on space continuity
At low scales, GR coordinates shouldn't be arbitrary smooth.

Quantizing GR allows corrections at low scale

Gravitating QM allows corrections at large scale

Non-Relativistic GUP

- The quadratic momenta corrections to Heisenberg uncertainty principle as suggested by Kempf-Mangano-Mann reads

$$[x_i, p_j] = i\hbar\left(\delta_{ij} + \frac{\beta_0}{(M_P c)^2}\delta_{ij}p^2 + \frac{2\beta_0}{(M_P c)^2}p_i p_j\right)$$

where $p^2 := \vec{p}^2 = \sum_{j=1}^{3} p^j p_j$ and $M_P = \sqrt{\hbar c/G}$ is the Planck mass.

and

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 + \frac{\beta_0}{(M_P c)^2}\left((\Delta p)^2 + \langle p\rangle^2 + 2\Delta p_i^2 + 2\langle p_i\rangle^2\right)\right)$$

implying a minimum measurable length

$$\Delta x_{min} = \sqrt{3\beta_0}\, L_P, \text{ where } L_P = \sqrt{\hbar G/c^3}$$

- There are various GUP proposals, for example, Maggiore,

$$[x_i, p_j] = i\hbar\,\delta_{ij}\sqrt{1 + \frac{\gamma_0}{(M_P c)^2}(p^2 + m^2c^2)},$$

motivated by quantum deformation of the Pioncarre algebra implying a minimal length associated with as

$$\Delta x_{min} \simeq \sqrt{\gamma_0/2}\, L_P$$

Need for Relativistic GUP

The non-relativistic 3d-GUP has no temporal dimension. Thus, in spacetime:

1. **either commutators or uncertainties are NOT necessarily Lorentz covariant, *PPNL13(2016)59***
2. **this means that Δx_{min} is frame dependent, and**
3. **this causes nonlinear additional law of momenta.**

With the Lorentz transformation represented by the unitary operator

$$U(p^{\nu}, M^{\rho\sigma}) = e^{i\alpha_{\nu}p_{\nu}} e^{\frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}}$$

we get $\acute{x}^{\mu} = U\, x^{\mu} U^{-1}$, $\acute{p}^{\mu} = U\, p^{\mu} U^{-1}$, and find that

$$[\acute{x}^{\mu}, \acute{p}^{\nu}] = i\hbar\left(1 + \beta \acute{p}^{\rho}\acute{p}_{\rho}\right)\eta^{\mu\nu} + i\hbar\beta\acute{p}^{\mu}\acute{p}^{\nu}$$

which is Lorentz covariant

Need for Relativistic GUP

- Non-relativistic 3d-GUP generalizes the momentum operator $\hat{p}_i = \hat{p}_{0i}\left(1 + \beta\, p^2\right)$ but not length operator $\hat{x}_i = \hat{x}_{0i}$,
- Accordingly, relativistic dispersion relation is deformed

  $$E^2 = \left(m\, c^2\right)^2 + (pc)^2 + \mathcal{O}\left[p^4\right],$$

  and nonlinear additional law of momenta, like $\hat{p}_3 = \hat{p}_2 + \hat{p}_1$ appears.

- In Poincare algebra, the generator of the Lorentz group

  $$M^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu = \left(1 + \beta p_0^\rho p_{0\rho}\right) \acute{M}^{\mu\nu}$$

  where $\acute{M}^{\mu\nu} = p_0^\mu x_0^\nu - p_0^\nu x_0^\mu$, $p_\mu p^\mu$ is Casimir operator of Lorentz algebra which commutes with ***p*** and p^2; $\left[M_{\mu\nu}, p^2\right] = \left[p^2, p_\mu\right] = 0$.

- The operator $p_\mu p^\mu$ also commutes with other Casimirs, like $W^\mu W_\mu$ where $W_\mu \coloneqq \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} p^\lambda$, so that $\left[p_\mu, W_\mu\right] = \left[p, W^\nu W_\nu\right] = 0$.

- This leads to $p_\mu p^\mu = -c^4 m^2$, squared dispersion relation and thus fulfilling linearity of additional law of momenta.

Need for Relativistic GUP

- **To assure that Δx_{min} is Lorentz invariant, we start with the spacetime noncommutativity**

$$[x^{\mu}, x^{\nu}] = -2i\hbar(x^{\mu}p^{\nu} - x^{\nu}p^{\mu})$$

and the length-momentum noncommutativity

$$[x^{\mu}, p^{\nu}] = (x^{\mu}p^{\nu} - p^{\nu}x^{\mu}) = i\hbar(\eta^{\mu\nu} + 2\beta p^{\mu}p^{\nu})$$

- **Then,**

$$[x^{\mu}, x^{\nu}] = 2\,\hbar^{2}(\eta^{\mu\nu} + 2\beta p^{\mu}p^{\nu}) - 2i\hbar M^{\nu\mu}$$

- **This means that the spacetime coordinates**
  1. **are likely noncommutative,**
  2. **have similarities with Snyder algebra but**
  3. **not forming a closed algebra (as depending on *p*)**

Relativistic GUP

- We assume that the physical position and momentum in terms of their auxiliary ***4-vectors*** x_0^{μ} and p_0^{μ}, read

  $\hat{x}^{\mu} = \hat{x}_0^{\mu} = (x_0^0, x_0^i) \qquad \hat{p}^{\mu} = \hat{p}_0^{\mu}(1 + \beta p_0^{\rho} p_{0\rho}) = (p_0^0, p_0^i)(1 + \beta p_0^{\rho} p_{0\rho})$

  where $i \in \{1, 2, 3\}$ and $\mu, \nu \in \{0, 1, 2, 3\}$, ρ is a dummy index, $x_0^0 = ct, p_0^0 = E/c$ are parameters, and $\hat{x}_0^{\mu}$, $\hat{p}_0^{\mu}$ are canonically conjugate variables, $[\hat{x}_0^{\mu}, \hat{p}_0^{\nu}] = i\hbar\eta^{\mu\nu}$.

- The relativistic generalized uncertainty principle is given as

$$[\hat{x}^{\mu}, \hat{p}^{\nu}] = i\hbar\left[\left(1 + \beta \hat{p}_0^{\rho} \hat{p}_{0\rho}\right)\eta^{\mu\nu} + 2\beta \hat{p}^{\mu} \hat{p}^{\nu}\right]$$

- From Robertson uncertainty principle which follows from Schrödinger uncertainty principle,

$$\Delta x^{\mu} \Delta p^{\nu} \geq \frac{1}{2} |\langle [\hat{x}^{\mu}, \hat{p}^{\nu}] \rangle|$$

- The relativistic generalized length-momentum uncertainties in curved spacetime, $\eta^{\mu\nu} \to g^{\mu\nu}$, are given as

$$\Delta x^{\mu} \Delta p^{\nu} \geq \frac{\hbar}{2}\left[g^{\mu\nu} + \beta(\Delta p)^2 + \beta\langle p \rangle^2 - \beta(\Delta p^{\mu})^2 - \beta(\Delta p^{\nu})^2\right]$$

Relativistic GUP

- To have real roots for Δp^ν, it is required that

$$(\Delta x^\mu)^2 \geq \hbar^2\left[\beta^2\left(\langle p\rangle^2 - (\Delta p^\mu)^2 - (\Delta p^\nu)^2\right) - \beta g^{\mu\nu}\right]$$

- This leads to a minimum measurable length, which is frame (coordinate) independent

$$\Delta x^\mu_{min} \geq \pm\sqrt{-g^{\mu\nu}}\,\hbar\sqrt{\beta} = \pm\sqrt{-|g|}\sqrt{\beta_0}\,\ell_p$$

- The result that the minimum measurable length Δx^μ_{min} is given in terms of $\sqrt{-det\ g}$ is very interesting, as in **GR** no physical dimensions are assigned to the coordinates. They are fundamentally arbitrary.

- Jacobian determinant ***J*** and $\sqrt{-det\ g}$ (sign) assure invariant transformation from one system of coordinates to another, so that

$$\sqrt{-det\ g}\,\Delta x^\mu_{min} = \Delta \acute{x}^\mu_{min}$$

is frame independent (Lorentz invariant).

Relativistic GUP

Now, we can determine the uncertainties in **GR**:

- For length

$$x^{\mu} = x_0^{\mu} - \Delta x_{min}^{\mu} = \left(\left(x_0^0 - |\Delta x_{min}| \right), \left(x_0^i - |\Delta x_{min}| \right) \right)$$

then

$$\Delta x^{\mu} = \Delta x_0^{\mu} - 2\sqrt{\beta_0} \ell_p$$

- For momentum, which follows from roots of

$$\Delta x^{\mu} \Delta p^{\nu} \geq \frac{\hbar}{2} \left[g^{\mu\nu} + \beta (\Delta p)^2 + \beta \langle p \rangle^2 - \beta (\Delta p^{\mu})^2 - \beta (\Delta p^{\nu})^2 \right]$$

then

$$\Delta p^{\nu} \leq \frac{1}{\hbar\beta} \left[\Delta x^{\mu} \pm \left((\Delta x^{\mu})^2 - \beta \hbar^2 g^{\mu\nu} \right)^{1/2} \right]$$

$$\Delta p^{\nu} \leq \frac{1}{\hbar\beta} \left[\Delta x^{\mu} \pm \left(\Delta x^{\mu} + \frac{1}{2} \frac{\left(\Delta x_{min}^{\mu} \right)^2}{\Delta x^{\mu}} \right) \right]$$

- This suggests that $-\frac{\Delta x^{\mu}}{\hbar\beta} \frac{1}{2} \left(\frac{\Delta x_{min}^{\mu}}{\Delta x^{\mu}} \right)^2 \leq \Delta p^{\nu} \leq 2 \frac{\Delta x^{\mu}}{\hbar\beta} \left[1 + \frac{1}{4} \left(\frac{\Delta x_{min}^{\mu}}{\Delta x^{\mu}} \right)^2 \right]$

$$\Delta p^{\nu} \leq \frac{2}{\hbar\beta} \Delta x^{\mu} \qquad \Delta p_0^{\nu} \leq \frac{2}{\hbar\beta} \frac{\Delta x_0^{\mu} - 2\sqrt{\beta_0} \ell_p}{1 + \beta g^{0\rho} \Delta |p_0|^2}$$

Relativistic GUP

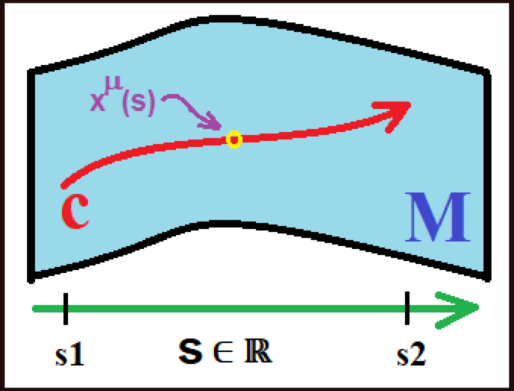
If these results were correct, at relativistic energy scale, we conclude that

- the spacetime is neither smooth nor continuous as an inaccessible spacetime element whose volume characterized by Δx^{μ}_{min} exists,
- not only coordinates and momenta are uncertain, but other physical quantities have noncommutative relations, as well,
- due uncertainties, events likely happen in jumps with nondeterministic outcomes,
- the measurements are likely neither precise nor noncoherent.

Generalized Manifold (Finsler)

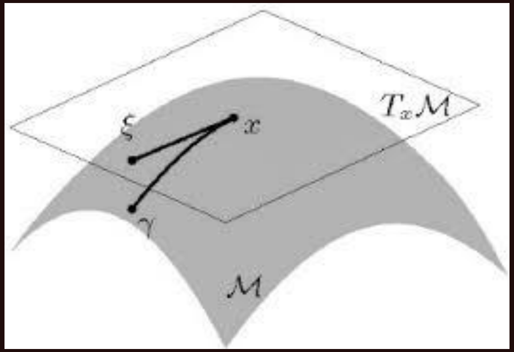
- **Generalization of $g^{\mu\nu}$ would be possible on generalized Riemann manifold *M*:**

- **Riemann geometry *(M,g)*: at point *x*:**
  - **metric tensor $g = g^{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}$,**
  - **length of curve *c*, $\int_c \sqrt{g^{\mu\nu}(x)dx^{\mu}dx^{\nu}}$ or $\int_{s_1}^{s_2} \sqrt{g^{\mu\nu}(s)d\dot{x}^{\mu}d\dot{x}^{\nu}}ds$, where $\dot{x} = \frac{dx}{ds}$**



- **Finsler geometry *(M,F)*: at x on *M*, the Finsler structure $F(x,\dot{x})$ is related to the generalized metric tensor**

$$F = \sqrt{g^{\mu\nu}(x)d\dot{x}^{\mu}d\dot{x}^{\nu}} \text{ and } g := \frac{1}{2}\frac{\partial^2 F^2}{\partial\dot{x}^{\mu}\partial\dot{x}^{\nu}}.$$



- **F is +ive for $\dot{x} \neq 0$ on tangent bundle *TM* & homogeneous of degree *1* in $\dot{x}$,**
- **therefore, on *TM*, at local coordinates $(x,\dot{x})$,**

$$F(x, \quad \dot{x}) = \lambda F(x,\dot{x}), \quad \forall \lambda \in R^+$$

- **the ratio of lengths of any two collinear vectors doesn't include metric functions.**

Generalized Manifold (Finsler)

- In Finsler geometry, the special case $F(x,\dot{x}) = \sqrt{g^{\mu\nu}(x)d\dot{x}^{\mu}d\dot{x}^{\nu}}$ distinguishes Finsler from Riemann geometry; a relaxation of quadratic restriction *Notices Amer. Math. Soc. 43(1996) 95*.
- The length of a curve ***c*** is given as $\int_c F(x,\dot{x})$ or $\int_{s1}^{s2} F\big(x^{\mu}(s),\dot{x}^{\mu}(s)\big)ds$.
- If Δx^{μ}_{min} sets limitations on the space and momentum coordinates in **GR** and determines their uncertainties, their measurements are likely no longer arbitrary precise or noncoherent. Thus, we assume that

$$F\big(x,\Delta x^{\mu}_{min}\dot{x}\big) = \Delta x^{\mu}_{min} F(x,\dot{x}), \quad \forall\, \Delta x^{\mu}_{min} \geq 0.$$

- On ***TM***, the metric tensor given as $g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}$ could be determined by the Hessian in the $(x,\dot{x})$-coordinates,

$$g_{AB} = \frac{1}{2}\frac{\partial^2 F^2\big(x,\Delta x^{\mu}_{min}\dot{x}\big)}{\partial \dot{x}^{\alpha}\partial \dot{x}^{\beta}}$$

where each $g_{\mu\nu}$ is homogeneous of degree ***0*** in $\dot{x}$.

Quantized Metric Tensor

This leads to

$$\tilde{g}_{\mu\nu} = g_{AB} \frac{\partial x^A}{\partial \xi^\alpha} \frac{\partial x^B}{\partial \xi^\beta}$$

where $A, B \in \{0, 1, 2, \dots, 7\}$, and $\alpha, \beta, \mu, \nu \in \{0, 1, 2, 3\}$. Then,

$$\tilde{g}_{\mu\nu} = \left[1 + \left(-|g|\hbar^2\beta|\ddot{x}|^2\right)\right] g_{\mu\nu} = \left[1 + \left(-|g|\beta_0 \ell_p^2 |\ddot{x}|^2\right)\right] g_{\mu\nu}$$

where $|\ddot{x}|^2 = \ddot{x}^\lambda \ddot{x}_\lambda = g_{\gamma\delta}\ddot{x}^\delta \ddot{x}^\gamma$ and λ, γ, δ are dummy indices.

To summarize, for quadratic ***F***, $g_{\mu\nu}$ reduces to $g_{\mu\nu}(x)$ living on ***M***.

- On ***TM***, coordinates are 8d, $x^A = (x^\alpha,\ \Delta x^\mu_{min} \dot{x}^\alpha)$,
- On ***TM***, the line element is given as $d\tilde{s}^2 = g_{AB} dx^A dx^B$,
- On ***M***, 4d-manifold, the line element is $d\tilde{s}^2 = \tilde{g}_{\mu\nu} d\xi^\mu d\xi^\nu$.

Quantized Metric Tensor

- On the Riemann manifold ***M***, $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \left(-|g|\beta_0 \ell_p^2\right)|\ddot{x}|^2\, g_{\mu\nu}$ or

$$\tilde{g}_{\mu\nu} = \left(1 + \mathcal{T}|\ddot{x}|^2\right) g_{\mu\nu}$$

where $\left(-|g|\beta_0 \ell_p^2\right)|\ddot{x}|^2\, g_{\mu\nu} = \left(1 + \mathcal{T}|\ddot{x}|^2\right) g_{\mu\nu}$ could be seen as local perturbations to the curved spacetime.

- Quantization of the local perturbations is possible through:
  - coordinate-independent discretization $\left(-|g|\beta_0 \ell_p^2\right)$, and
  - second-order derivatives of the coordinates $|\ddot{x}|^2$.

- $|\ddot{x}|^2 = \ddot{x}^\lambda \ddot{x}_\lambda = \frac{\partial \dot{x}^\lambda}{\partial \xi^\lambda} \frac{\partial \dot{x}_\lambda}{\partial \xi_\lambda}$ could be treated as
  - spacelike four-acceleration, or
  - local geodesic equation.

- To count for consequences of Δx^μ_{min}, tangent bundle ***TM*** and Finsler manifold with quadratic ***F*** are assumed, on which $g_{\mu\nu}$ reduces to the usual $g_{\mu\nu}(x)$ living on ***M***.

Stress-Energy Tensor

In curved space , the full action consists of the Einstein-Hilbert action and the non-gravitational part of the Lagrangian density

$$S = \int \frac{c^4}{16\pi G}\left(R + \mathcal{L}_{\mathtt{matter}}\right)\sqrt{-g}\, d^4x,$$

The physical quantities including the Hilbert stress-energy tensor can be determined from demanding vanishing variation of this action with respect to the inverse metric tensor:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-|g|}}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{-|g|}\mathcal{L}_{\mathtt{matter}}\right) = -2\frac{\partial \mathcal{L}_{\mathtt{matter}}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\mathtt{matter}}$$

$T_{\mu\nu}$ is characterized by the metric tensor and the matter field, only.

$$\tilde{T}_{\mu\nu} = -2\frac{\partial \tilde{\mathcal{L}}_{\mathtt{matter}}}{\partial \tilde{g}^{\mu\nu}} + \tilde{g}_{\mu\nu}\tilde{\mathcal{L}}_{\mathtt{matter}}$$

$$\tilde{g}_{\mu\nu} = \left(1 + \mathcal{T}|\ddot{x}|^2\right)g_{\mu\nu}$$

$$\delta\tilde{g}^{\mu\nu} = \left(1+\mathscr{T}|\ddot{x}|^2\right)^{-1}\delta g^{\mu\nu} - \left(1+\mathscr{T}|\ddot{x}|^2\right)^{-2}\mathscr{T}\left[|\ddot{x}|^2\delta g^{\mu\nu} + g^{\mu\nu}\ddot{x}^{\mu}\delta\ddot{x}_{\mu} + g^{\mu\nu}\ddot{x}^{\nu}\delta\ddot{x}_{\nu}\right]$$

Stress-Energy Tensor

$$\tilde{T}_{\mu\nu} = \frac{-2\left(1+\mathscr{T}|\ddot{x}|^2\right)^2}{1-\mathscr{T}g^{\mu\nu}\left(\ddot{x}^{\mu}\frac{\partial \ddot{x}_{\mu}}{\partial g^{\mu\nu}}+\ddot{x}^{\nu}\frac{\partial \ddot{x}_{\nu}}{\partial g^{\mu\nu}}\right)}\frac{\partial \tilde{\mathcal{L}}_M}{\partial g^{\mu\nu}} + \left(1+\mathscr{T}|\ddot{x}|^2\right)g_{\mu\nu}\tilde{\mathcal{L}}_M$$

The Lagrangian density can be generalized as

$$\mathcal{L}(q^i,\dot{q}^i) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j + V(q^i)$$

Electromagnetic field in curved spacetime

$$\tilde{\mathcal{L}}_{\rm EM} = -\frac{1}{4}\tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = -\frac{1}{4}\left(1+\mathscr{T}|\ddot{x}|^2\right)^{-2}g^{\mu\nu}g^{\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \left(1+\mathscr{T}|\ddot{x}|^2\right)^{-2}\mathcal{L}_{EM}$$

Quantum-induced corrections are linearly factorized

$|\ddot{x}|^2\ [\mathscr{T}]$

Klein Gordon field (scalar field) in curved spacetime

$$\tilde{\mathcal{L}}_{\phi} = \frac{1}{1+\mathscr{T}|\ddot{x}|^2}\left[\mathcal{L}_{\phi} - \mathscr{T}|\ddot{x}|^2 V(\phi)\right]$$

Quantum-induced corrections are linearly factorized

Stress-Energy Tensor

The stress-energy tensor with EM Lagrangian density

$$\tilde{T}_{\mu\nu} = \left[1 - \mathscr{T} g^{\mu\nu}\left(\ddot{x}^{\mu}\frac{\partial \ddot{x}_{\mu}}{\partial g^{\mu\nu}} + \ddot{x}^{\nu}\frac{\partial \ddot{x}_{\nu}}{\partial g^{\mu\nu}}\right)\right]^{-1} \left\{T_{\mu\nu} - \left(1+\mathscr{T}|\ddot{x}|^2\right)\mathscr{T}\left[|\ddot{x}|^2 g_{\mu\nu} - 4\frac{|\ddot{x}|^2}{g_{\mu\nu}} - 3\left(\ddot{x}^{\mu}\frac{\partial \ddot{x}_{\mu}}{\partial g^{\mu\nu}} + \ddot{x}^{\nu}\frac{\partial \ddot{x}_{\nu}}{\partial g^{\mu\nu}}\right)\right]\mathcal{L}_{\text{EM}}\right\}$$

Quantum-induced corrections are linearly factorized to classical $T^{\mu\nu}$

Covariant derivative of stress-energy tensor with EM Lagrangian

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = \left\{\left[\frac{4\mathscr{T}|\ddot{x}|^2\eta_{\mu\nu} + \eta_{\mu\nu}}{\left(1+\mathscr{T}|\ddot{x}|^2\right)}\right] - \eta_{\mu\nu}\right\}\left[-\frac{1}{4}\left(F_{\alpha\beta}F_{\mu\nu,\mu} + F_{\mu\nu}F_{\alpha\beta,\mu}\right)\right]$$

- for unquantized stress–energy tensor, i.e., vanishing β_0 and/or $|\ddot{x}|^2$,

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = 0, \qquad (44)$$

- for quantized stress–energy tensor, i.e., finite β_0 and/or $|\ddot{x}|^2$, in vacuum spacetime, i.e., vanishing four–current,

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = 0, \qquad (45)$$

as $F_{\mu\nu,\mu} = F_{\alpha\beta,\mu} = 0$. Otherwise, $\nabla^{\mu}\tilde{T}_{\mu\nu}$ is divergent! In that case, the Lagrangian density should be coupled to charged particle as source in the inhomogeneous Maxwell's equations so that

$$\nabla^{\mu}\left(\tilde{T}^{\text{EM}}_{\mu\nu} + \tilde{T}^{\text{particle}}_{\mu\nu}\right) = 0, \qquad (46)$$

Stress-Energy Tensor

The stress-energy tensor with scalar Lagrangian density

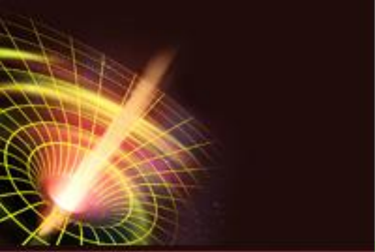
$$\tilde{T}_{\nu\mu} = -\frac{2\left(1+\mathscr{T}|\ddot{x}|^2\right)^2}{1-\mathscr{T} g^{\nu\mu}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right)} \frac{\partial \mathcal{L}_\phi}{\partial g^{\nu\mu}} + g_{\nu\mu}\mathcal{L}_\phi - g_{\nu\mu}\mathscr{T}|\ddot{x}|^2 V(\phi) - \frac{1}{2}\frac{\mathscr{T}\left(1+\mathscr{T}|\ddot{x}|^2\right)^{-2}|\ddot{x}|^2 + \mathscr{T}|\ddot{x}|^2\left(1+\mathscr{T}|\ddot{x}|^2\right)^{-2}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right)}{1-\mathscr{T} g^{\nu\mu}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right)} g^{\nu\mu}\nabla_\nu\phi\nabla_\mu\phi.$$

Quantum-induced corrections are linearly factorized to classical $T^{\mu\nu}$

Covariant derivative of stress-energy tensor with EM Lagrangian

$$\nabla^\mu \tilde{T}_{\mu\nu} = 0.$$

We conclude that the stress-energy tensor is not explicitly depending on the coordinates x^μ, i.e., it is locally conserved.



Summary

RGUP

$$\hat{x}^\mu = \hat{x}_0^\mu = (x_0^0, x_0^i) \quad \text{and} \quad \hat{p}^\mu = \hat{p}_0^\mu(1+\beta p_0^\rho p_{0\rho}) = (p_0^0, p_0^i)(1+\beta p_0^\rho p_{0\rho}),$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar[(1+\beta\hat{p}_0^\rho \hat{p}_{0\rho})\eta^{\mu\nu} + 2\beta\hat{p}^\mu\hat{p}^\nu],$$

$$\Delta x^\mu \Delta p^\nu \geq \frac{\hbar}{2}[g^{\mu\nu} + \beta(\Delta p)^2 + \beta\langle p\rangle^2 - \beta(\Delta p^\mu)^2 - \beta(\Delta p^\nu)^2],$$

$$\Delta x^\mu_{min} \geq \pm\sqrt{-g^{\mu\nu}}\,\hbar\sqrt{\beta} = \pm\sqrt{|g|}\sqrt{\beta_0}\,\ell_p \quad \text{and} \quad \Delta x^\mu = \Delta x_0^\mu - 2\sqrt{\beta_0}\,\ell_p,$$

$$\Delta p^\nu \leq \frac{1}{\hbar\beta}\left[\Delta x^\mu \pm \left(\Delta x^\mu + \frac{1}{2}\frac{(\Delta x^\mu_{min})^2}{\Delta x^\mu}\right)\right] \quad \text{or} \quad \Delta p^\nu \leq \frac{2}{\hbar\beta}\Delta x^\mu \quad \text{and} \quad \Delta p_0^\nu \leq \frac{2}{\hbar\beta}\frac{\Delta x_0^\mu - 2\sqrt{\beta_0}\,\ell_p}{1+\beta g^{0\rho}\Delta|p_0|^2}.$$

Metric tensor

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + (-|g|\beta_0\ell_p^2)|\ddot{x}|^2\, g_{\mu\nu} \qquad \text{where } |\ddot{x}|^2 = \ddot{x}^\lambda\ddot{x}_\lambda = \frac{\partial\dot{x}^\lambda}{\partial\xi^\lambda}\frac{\partial\dot{x}_\lambda}{\partial\xi_\lambda}$$

Quantum-induced corrections are linearly added so that vanishing corrections retrieve GR metric tensor $g_{\mu\nu}$.

Stress-Energy Tensor

$$\tilde{T}_{\mu\nu} = \left[1 - \mathscr{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]^{-1} \left\{T_{\mu\nu} - (1+\mathscr{T}|\ddot{x}|^2)\,\mathscr{T}\left[|\ddot{x}|^2 g_{\mu\nu} - 4\frac{|\ddot{x}|^2}{g_{\mu\nu}} - 3\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]\mathcal{L}_{\text{EM}}\right\}$$

Quantum-induced corrections are linearly factorized to the classical $T^{\mu\nu}$ so that vanishing corrections retrieve GR $T^{\mu\nu}$.


Thank you!