Evolution of Cluster of Stars in $f(R)$ Gravity

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Layout of Talk

- **Definitions**
- **Evolution of Cluster of Stars in $f(R)$ Gravity**

Cluster of Stars

- **Star clusters are groups of hundreds to millions of stars that provide astronomers crucial insight into stellar evolution through comparisons of stars' ages and compositions.**
- **Star clusters form out of large interstellar regions of gas and dust called molecular clouds.**
- **The densest areas of these molecular clouds collapse into themselves to form stars.**

Cluster of Stars

- **In some cases, the stars disperse after their creation.**
- **However, if there are enough stars formed close enough together, they may remain gravitationally bound and live as a star cluster.**
- **But star clusters are not galaxies which, confusingly, are also gravitationally bound groups of stars.**
- **We do not understand well enough if any clear division between star cluster and galaxies actually exists.**

Cluster of Stars

- **But the most straightforward differentiation would be whether a group of stars is bound together by their own gravity or if you need to have dark matter included in that cluster in order to keep it together.**
- **If there's dark matter in the mix, then that group of stars is likely a galaxy.**
- **Stars within a galaxy, can be a variety of ages and have a variety of compositions.**
- **Galaxies are like the cities that star clusters live in. Galaxies can contain thousands or more star clusters, and many molecular clouds, and dark matter.**
- **That's why stellar clusters are so important to astronomers who are studying stellar evolution.**

Dark Matter

- Dark matter is a kind of hidden matter which acts like a gravitational pull to our universe.
- An undetectable form of mass that emits no light or no photons, but we know it must exist because we observe the effects of its gravity.
- The problems of galactic clusters, galactic rotational curves and the mass discrepancy indicate the importance and existence of dark matter.
- Different detection mechanism have been constructed to study the nature of dark matter.
- Among these, the search for appearing of dark matter in stellar bodies comes as inexpensive and ingenious detection strategy.

- Useful Candidate for dark energy and dark matter
- The $f(R)$ theory is followed by the following action [Buchdahl, H. A. (1970); Monthly Notices of the Royal Astronomical Society.150]

$$S_f(R) = \frac{1}{\kappa}\int d^4x\sqrt{-g}f(R) + S_M. \tag{1}$$

Here κ shows the coupling constant, S_M represents the matter related segment and $f(R)$ describes a non-linear Ricci scalar part.

- In this context, Starobinskyy [Phys. Lett. **B91**(1980)99] used second order curvature term,
  $f(R) = R + \epsilon R^2$,
  to formulate Einstein equations in relation to quantum one loop distribution. He used this model to describe cosmic exponential expansion and present time cosmic acceleration for early time as well as present time of power-law inflation.

- Moreover, It has been shown that this model can be used for dark matter problem according to WMAP date for $\varepsilon = \frac{1}{6M^2}$ with $M = 2.7 \times 10^{-12}\ GeV,\ (\varepsilon = 2.3 \times 10^{22} Ge/V^2)$. The square-order curvature term behave like a scalar degree of freedom (scalar graviton) whose interaction with standard model particles yields abundance of thermal energy which provide dark matter regions. The scalar graviton of mass $M < 10^{-12}$ is responsible for Yukawa force of attraction between dark matter particles of dissimilar masses [Cembranos, J. A. R. et al.: J. Cosmol. Astropart. Phys. **4**(2012)021]

- It is well-known that star cluster are self-gravitating in nature.
- The process of self-gravitation is one of the principal features of astrophysics which controls the existence of stellar structures in the universe.
- The constituent parts of these bodies are held together under the influence of their own gravity.
- In the absence of self-gravitation, all celestial bodies (like stars, galaxies and clusters etc.) will expand and totally dissipate.

Evolution of Cluster of Stars in $f(R)$ Gravity

- Herrera et al. explored the evolution of spherically symmetric relativistic self-gravitating fluid distribution by considering the structure scalars with homogenous expansion and homologous evolution.[Phys. Rev. D98, 104059, (2018)]
- It is interesting to explore of evolution of cluster of stars in the presence of dark matter by considering star cluster as self-gravitating system in $f(R)$ gravity.

The metric $f(R)$ field equations are obtained by vary (1) with respect to $g_{\iota\omega}$ as follows

$$R_{\alpha\beta}f(R)-\frac{1}{2}f(R)g_{\alpha\beta}+(g_{\alpha\beta}\Box-\nabla_{\alpha}\nabla_{\beta})f_R=\omega T_{\alpha\beta}. \tag{2}$$

Here $R_{\alpha\beta}$ express Ricci tensor, $f_R=\frac{df}{dR}$, and $T_{\alpha\beta}$ shows ideal energy momentum tensor of matter distribution which is given by

$$T_{\alpha\beta}=-\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}. \tag{3}$$

In the form of Einstein tensor, Eq (2) can be composed as

$$G_{\alpha\beta}=\kappa T^{(eff)}_{\alpha\beta}=\frac{\kappa}{f_R}[(T_{\alpha\beta}+T^{(D)}_{\alpha\beta})]. \tag{4}$$

Here

$$T^{(D)}_{\alpha\beta}=\frac{1}{\kappa}\nabla_{\alpha}\nabla_{\beta}f_R-\Box f_R g_{\alpha\beta}+(f-Rf_R)\frac{g_{\alpha\beta}}{2}, \tag{5}$$

where superscript D represents dark source parts (may be DE or DM).

Cluster of Stars

We assume an evolving cluster of stars in the form of spherically symmetric fluid distribution where the fluid particles is a combination of baryonic matter (stars of cluster) and non-baryonic part (dark matter). The collection of stars is represented by a spherically symmetric line element

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)dr^2 + C^2(t,r)d\theta^2 + C^2(t,r)sin^2\theta d\phi^2. \tag{6}$$

The canonical form of energy momentum tensor appropriated as

$$T_{\alpha\beta}^{(eff)} = \mu^{(eff)}V_\alpha V_\beta + Ph_{\alpha\beta}^{(eff)} + \Pi_{\alpha\beta}^{(eff)} + q^{(eff)}(V_\alpha\chi_\beta + \chi_\alpha V_\beta), \tag{7}$$

where

$$P = \frac{P_r^{(eff)} + 2P_\perp^{(eff)}}{3}, \quad h_{\alpha\beta} = g_{\alpha\beta} + V_{\alpha V_\beta},$$

$$\Pi_{\alpha\beta}^{(eff)} = \Pi^{(eff)}(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}), \quad \Pi^{(eff)} = P_r^{(eff)} - P_\perp^{(eff)}.$$

The field equations with dark source terms are as follows

$$
\begin{aligned}
G_{00}=\mu^{(eff)} &= \frac{1}{1+2\epsilon R}\left[\mu^{m}A^{2}\kappa+A^{2}\left(\frac{-\epsilon R^{2}}{2}\right)-\frac{2\epsilon A^{2}R''}{B^{2}}\right.\\
&+ \left.\left(\frac{2\dot{C}}{C}-\frac{\dot{B}}{B}\right)2\epsilon\dot{R}+\left(\frac{2C'}{C}-\frac{B'}{B}\right)\frac{2\epsilon A^{2}R'}{B^{2}}\right], \qquad (8)\\
G_{01}=q^{(eff)} &= \frac{8\pi}{R+\varepsilon R^{2}}\left[\frac{2\varepsilon A'}{A}(C'+\dot{C})+\frac{2\varepsilon C'}{B}(B'-\dot{B})\right.\\
&- \left.2\varepsilon C'^{\cdot}+kABq\right], \qquad (9)\\
G_{11}=P_{r}^{(eff)} &= \frac{1}{1+2\epsilon R}\left[P_{r}^{m}B^{2}\kappa-B^{2}\left(\frac{-\epsilon R^{2}}{2}\right)+\frac{2\epsilon B^{2}\ddot{R}}{A^{2}}\right.\\
&- \left.\left(\frac{2\dot{C}}{C}+\frac{\dot{A}}{A}\right)\frac{2\epsilon B^{2}\ddot{R}}{A^{2}}-2\epsilon R'\left(\frac{2C'}{C}+\frac{A'}{A}\right)\right], \qquad (10)\\
G_{22}=P_{\perp}^{(eff)} &= \frac{1}{1+2\epsilon R}\left[P_{\perp}^{m}C^{2}\kappa-C^{2}\left(\frac{-\epsilon R^{2}}{2}\right)-\frac{2\epsilon C^{2}R''}{B^{2}}\right.\\
&+ \frac{2\epsilon C^{2}\ddot{R}}{A^{2}}-\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)\frac{2\epsilon C^{2}\dot{R}}{A^{2}}\\
&- \left.\left(\frac{2C'}{C}+\frac{A'}{A}\right)\frac{2\epsilon C^{2}R'}{B^{2}}\right]. \qquad (11)
\end{aligned}
$$

The acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta}V^\beta, \quad \Theta = V^\alpha_{;\alpha}. \tag{12}$$

The shear tensor $\sigma_{\alpha\beta}$ is

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \tag{13}$$

By using (46) with (12), the obtain expression of non-vanishing component of four-acceleration and its scalar is

$$a_1 = A^{-1}A', \quad a = \sqrt{a^\alpha a_\alpha} = \frac{A'}{AB}, \tag{14}$$

and the expansion scalar is calculated as

$$\Theta = A^{-1}(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}). \tag{15}$$

Here prime and dot represent the differentiation with respect to r and t, respectively.

we obtain the non-zero components of shear tensor and shear scalar given by

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \sigma_{22} = \frac{\sigma_{33}}{sin^2\theta} = -\frac{1}{3}C^2\sigma \quad \sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2}{3}\sigma^2, \tag{16}$$

where

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right). \tag{17}$$

The Misner Sharp mass function $m(t, r)$ describes the total energy of the spherical system within radius "C" and reads as

$$m = \frac{C^3}{2}R^{23}_{23} = \frac{C}{2}\left[\left(\frac{\dot{C}}{A}\right)^2 - (\frac{C\prime}{B})^2 + 1\right]. \tag{18}$$

The proper time derivative D_T is given by

$$D_T = \frac{1}{A}\frac{\partial}{\partial t} \tag{19}$$

and the variation of areal radius C (inside the fluid) with respect to proper time provide the velocity of collapsing fluid given by

$$U = D_T C < 0. \tag{20}$$

Now, equation (18) can be rewritten as

$$E \equiv \frac{C'}{B} = \left(1 + U^2 - \frac{2m}{C}\right)^{\frac{1}{2}} \tag{21}$$

using (15), (17) and (21), Eq.(9) can be rewritten as

$$4\pi q^{(eff)} = E\left[\frac{1}{3}D_C(\Theta - \sigma) - \frac{\sigma}{C}\right], \tag{22}$$

where D_C denotes the proper radial derivative given by

$$D_C = \frac{1}{C'}\frac{\partial}{\partial r}. \tag{23}$$

Using field equations (8)-(9), with (19) and (23), we get from (18)

$$D_T m = -4\pi \left(P_r^{(eff)} U + q^{(eff)} E\right) C^2, \tag{24}$$

$$D_C m = 4\pi \left(\mu^{(eff)} + q^{(eff)} \frac{U}{E}\right) C^2, \tag{25}$$

which yield

$$m = 4\pi \int_0^r \left(\mu^{(eff)} + q^{(eff)} \frac{U}{E}\right) C^2 C' dr. \tag{26}$$

By using condition $m(t,0)=0$ and then Integrating gives

$$\frac{3m}{C^3}=4\pi\mu^{(eff)}-\frac{4\pi}{C^3}\int_0^r C^3\left(D_C\mu^{(eff)}-3q^{(eff)}\frac{U}{CE}\right)C'dr. \tag{27}$$

The Weyl tensor $C_{\alpha\mu\beta\nu}$ describes the effect of tidal forces and evaluated in terms of Riemann tensor, Ricci tensor and Ricci scalar. For spherically symmetric distribution the magnetic part of the Weyl tensor vanishes and the electric part of Weyl tensor $E_{\alpha\beta}$ [Herrera, L. et al.: Phys. Rev. D **69**(2004)084026] is given by

$$C_{\alpha\mu\beta\nu}=(g_{\alpha\mu\gamma\vartheta}g_{\beta\nu\delta\sigma}-\eta_{\alpha\mu\gamma\vartheta}\eta_{\beta\nu\delta\sigma})V^\gamma V^\delta E^{\vartheta\sigma},\quad g_{\alpha\mu\gamma\vartheta}=g_{\alpha\gamma}g_{\mu\vartheta}-g_{\alpha\vartheta}g_{\gamma\mu}, \tag{28}$$

where $\eta_{\alpha\mu\gamma\vartheta}$ is Levi-Civita tensor. Equation (28) implies

$$E_{\alpha\beta}=C_{\alpha\mu\beta\nu}V^\mu V^\nu. \tag{29}$$

The non-vanishing components of $E_{\alpha\beta}$ are

$$E_{11}=\frac{2}{3}B^2\varepsilon,\quad E_{22}=-\frac{1}{3}R^2\varepsilon,\quad E_{33}=E_{22}Sin^2\theta \tag{30}$$

with

$$\begin{aligned}\varepsilon \;=\; & \frac{1}{2A^2}\left[\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}-\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)\right]+\frac{1}{2B^2}[\frac{A''}{A}-\frac{C''}{C}+\left(\frac{B'}{B}+\frac{C'}{C}\right)\\ & \left(\frac{C'}{C}-\frac{A'}{A}\right)]-\frac{1}{2C^2}.\end{aligned} \tag{31}$$

The electric part of Weyl tensor can be also expressed as

$$E_{\alpha\beta}=\varepsilon\left(\chi_\alpha\chi_\beta-\frac{1}{3}h_{\alpha\beta}\right). \tag{32}$$

The Exterior Space time

The junction condition satisfied, if we assume bounded fluid distribution. Thus, outside σ we consider Vaidya spacetime (i.e radiations have no mass which are going outside)

$$ds^2 = -\left[1 - \frac{2M(v)}{r}\right] dv^2 - 2drdv + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{33}$$

By comparing the full non-adiabatic sphere to the Vaidya spacetime, when $r = r_\Sigma$ becomes constant require

$$m(t, r) = M(v), \tag{34}$$

and

$$2\left(\frac{\dot{C}'}{C} - \frac{\dot{B}}{B}\frac{C'}{C} - \frac{\dot{C}}{C}\frac{A'}{A}\right) = -\frac{B}{A}\left[2\frac{\ddot{C}}{C} - \left(2\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right)\frac{\dot{C}}{C}\right] + \frac{A}{B}[\left(2\frac{A'}{A} + \frac{C'}{C}\right) \frac{C'}{C} - \left(\frac{B}{C}\right)^2]. \tag{35}$$

Structure Scalars The structure scalars are scalar functions that are derived through the concept of orthogonal splitting of Riemann tensor [Herrera, L. et al.: Phys. Rev. D **79**(2009)064025.] and are used to explore dynamics of systems involving self-gravitating bodies. These factors suggest refine mechanism in stellar evolution by reducing complexity of the system . In order to suggest the dynamics of refine (complex less) evolution of collection of stars, we use the generalized concept of structure scalars in $f(R)$ gravity.

According to orthogonal splitting of the Reimann tensor, the tensors $Y_{\alpha\beta}$ and $X_{\alpha\beta}$ are given by

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}V^{\gamma}V^{\delta}, \tag{36}$$

$$X_{\alpha\beta} =^{*} R^{*}_{\alpha\gamma\beta\delta}V^{\gamma}V^{\delta} = \frac{1}{2}\eta^{\epsilon\rho}_{\alpha\gamma}R^{*}_{\epsilon\rho\beta\delta}V^{\gamma}V^{\lambda}, \tag{37}$$

where

$$R^{*}_{\alpha\gamma\beta\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R^{\epsilon\rho}_{\alpha\beta}. \tag{38}$$

The technique of decomposition of tensor into trace and tracefree part along with above define tensors provide some scalars quantities. These scalars are $Y_T^{(eff)}, Y_{TF}^{(eff)}, X_T^{(eff)}, X_{TF}^{(eff)}$ given by

$$Y_{\alpha\beta} = \frac{1}{3}Y_T^{(eff)}h_{\alpha\beta} + Y_{TF}^{(eff)}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{39}$$

$$X_{\alpha\beta} = \frac{1}{3}X_T^{(eff)}h_{\alpha\beta} + X_{TF}^{(eff)}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right). \tag{40}$$

By using orthogonal Splitting of Reimann tensor, we get By using $f(R)$ field equations and (31) in Eqs. (36) as well as (37), we get

$$Y_T^{(eff)} = 4\pi(\mu^{(eff)} + 3P_r^{(eff)} - 2\Pi^{(eff)}), \quad Y_{TF}^{(eff)} = \varepsilon - 4\pi\Pi^{(eff)}, \tag{41}$$

$$X_T^{(eff)} = 8\pi\mu^{(eff)}, \quad X_{TF}^{(eff)} = -\varepsilon - 4\pi\Pi^{(eff)}. \tag{42}$$

Equations (8), (10), (18) and (31) provide

$$\frac{3m}{R^3} = 4\pi(\mu^{(eff)} - \Pi^{(eff)}) - \varepsilon. \tag{43}$$

which combined with (27) and (41) give

$$Y_{TF}^{(eff)} = -8\pi\Pi + \frac{4\pi}{R^3}\int_0^r R^3\left(D_R\mu^{(eff)} - 3q^{(eff)}\frac{U}{RE}\right)R'dr. \tag{44}$$

Similarly, the scalar X_{TF} is obtained as

$$X_{TF}^{(eff)} = -\frac{4\pi}{R^3}\int_0^r R^3\left(D_R\mu^{(eff)} - 3q^{(eff)}\frac{U}{RE}\right)R'dr. \tag{45}$$

Structure Scalars as Evolution Parameters

Here, we discuss how a structure scalar can works as evolution parameter. From Eqs.(41)-(45) we evaluate the following results

$$X_T^{(eff)} = X_T^m + X_T^D = 8\pi\mu^{(eff)} = 8\pi(\mu^m + \mu^D), \tag{46}$$

$$X_{TF}^{(eff)} + Y_{TF}^{(eff)} = (X_{TF}^m + Y_{TF}^m) + (X_{TF}^D + Y_{TF}^D)$$
$$= -8\pi\Pi^{eff} = -8\pi(\Pi^m + \Pi^D), \tag{47}$$

$$X_{TF}^{(eff)} - Y_{TF}^{(eff)} = 2\varepsilon, \tag{48}$$

$$Y_T^{(eff)} - \frac{X_T^{(eff)}}{2} + X_{TF}^{(eff)} + Y_{TF}^{(eff)} = 4\pi(3P_r^{(eff)})$$
$$= 12\pi(P_r^m + P_r^D). \tag{49}$$

Starobinsky Model

For the following model

$$f(R) = R + \varepsilon R^2 \tag{50}$$

the field equations turn out to be

$$\mu^{(eff)} = \left[\frac{-A^2 f}{2} - \left(R'' - \frac{B\dot{B}\dot{R}}{A^2} - \frac{2B'R'}{B} - B\dot{B}\dot{R}\right) 2\epsilon + KA^2\mu\right] \times \frac{1}{R(1+\epsilon R)} + \frac{A^2 R}{2}, \tag{51}$$

$$P_r^{(eff)} = \left[\frac{B^2 f}{2} - \left(R'' - \frac{2B\dot{B}\dot{R}}{A^2} - \frac{2B'R'}{B}\right) 2\epsilon + KB^2 P_r\right] \times \frac{1}{R(1+\epsilon R)} - \frac{B^2 R}{2}, \tag{52}$$

$$P_\perp^{(eff)} = \left[\frac{R^2 f}{2} + \left(\frac{R\dot{R}^2}{A^2} - \frac{RR'^2}{B^2} - \frac{R\dot{R}^2}{A^2}\right) 2\epsilon + KP_\perp R^2\right] \times \frac{1}{R(1+\epsilon R)} + \frac{2\epsilon RR'^2}{B^2} - \frac{R^3}{2}, \tag{53}$$

$$q^{(eff)} = \left[\left(\dot{R}' - \frac{A'\dot{R}}{A} - \frac{\dot{B}R'}{B} - \frac{2A'\dot{R}}{A} - \frac{\dot{B}R'}{B}\right) 2\epsilon + KABq\right] \frac{1}{R(1+\epsilon R)}. \tag{54}$$

Homogeneous Density and Isotropic Case

The effective energy-momentum $T^{(eff)}_{\mu\nu}$ along with the field equations given as

$$C^{\eta}_{\alpha\beta\kappa;\eta} = T^{(eff)}_{\kappa[\alpha;\beta]} - \frac{1}{6} g_{\kappa[\alpha} T^{(eff)}_{,\beta]}. \tag{55}$$

Equation (28) implies

$$\mu^{\beta} C^{\eta}_{\alpha\beta\kappa;\eta} + \mu^{\beta}_{;\eta} C^{\eta}_{\alpha\beta\kappa;\eta} = \theta E_{\alpha\kappa} + u^{\mu} E_{\alpha\kappa;\mu} - u_{\kappa;\eta} E^{\eta}_{\alpha} - u_{\kappa} E^{\eta}_{\alpha;\eta},$$

which on contraction with $h^{\alpha}_{\mu} h^{\kappa}_{\nu} u^{\beta} \chi^{\mu} \chi^{\nu}$ gives

$$\begin{aligned} h^{\alpha}_{\mu} h^{\kappa}_{\nu} \mu^{\beta} \chi^{\mu} \chi^{\nu} C^{\eta}_{\alpha\beta\kappa;\eta} &= \frac{4\theta}{3} E_{\mu\nu} \chi^{\mu} \chi^{\nu} - u_{\nu;\eta} E^{\eta}_{\mu} \chi^{\mu} \chi^{\nu} + u^{\beta} E_{\alpha\kappa;\beta} h^{\alpha}_{\mu} h^{\kappa}_{\nu} \chi^{\mu} \chi^{\nu} \\ &+ h_{\mu\nu} \sigma^{\kappa\beta} E_{\kappa\beta} \chi^{\mu} \chi^{\nu} - \sigma_{\kappa\mu} E^{\kappa}_{\nu} \chi^{\mu} \chi^{\nu} - \sigma^{\kappa\nu} E^{\kappa}_{\mu} \chi^{\mu} \chi^{\nu}. \end{aligned} \tag{56}$$

The effective energy-momentum tensor becomes

$$h_\mu^\alpha h_\nu^\kappa u^\beta T^{(eff)}_{\kappa\alpha;\beta} = \Big[(P_\perp^{(eff)})_{;\beta} u^\beta h_{\mu\nu} + (\Pi^{(eff)}\chi_\kappa\chi_\alpha)_{;\beta} u^\beta h_\nu^\kappa h_\mu^\alpha$$

$$+q_\mu^{(eff)} a_\nu + q_\nu^{(eff)} a_\mu\Big], \tag{57}$$

$$h_\mu^\alpha h_\nu^\kappa u^\beta T^{(eff)}_{\kappa\beta;\alpha} = \left[(-\mu^{(eff)} + 3P_\perp^{(eff)})\left(\sigma_{\mu\nu} + \frac{\Theta h_{\mu\nu}}{3}\right) - q^{(eff)}_{;\alpha} h_\mu^\alpha \chi_\nu\right.$$

$$\left.+\Pi^{(eff)}_{;\beta}\mu^\beta \chi_\kappa \chi_{\beta;\alpha} u^\beta h_\mu^\alpha h_\nu^\kappa\right], \tag{58}$$

$$h_\mu^\alpha h_\nu^\kappa u^\beta g_{[\alpha} T^{(eff)}_{;\beta]} = h_{\mu\nu}[-\mu^{(eff)} + 3P_\perp^{(eff)} + \Pi^{(eff)}]_{;\beta} u^\beta. \tag{59}$$

From the Eqs. (55) and (57)-(59), we get

$$\left[\left(\varepsilon + \frac{4\pi\mu^{(eff)}}{2} - \frac{4\pi\Pi^{(eff)}}{2} -\right)_{,\mu} u^\mu + \frac{12\pi q^{(eff)} C'}{2BC} + \left(\varepsilon + \frac{4\pi\mu^{(eff)}}{2}\right.\right.$$

$$\left.\left.-\frac{4\pi\Pi^{(eff)}}{2} + 4\pi P_r^{(eff)}\right)(\Theta + \sigma)\right] = 0. \tag{60}$$

Similarly contraction of Weyl tensor and Ricci tensor with $u^\kappa u^\beta h^\alpha_\mu \chi^\mu$ yields

$$\left(\varepsilon + \frac{4\pi\mu^{(eff)}}{2} - \frac{4\pi\Pi^{(eff)}}{2}\right)_{,\mu}\chi^\mu - \frac{3C'}{BC}\left(\frac{4\pi\Pi^{(eff)}}{2} - \varepsilon\right)$$
$$-4\pi q^{(eff)}(\sigma + \Theta) = 0. \quad (61)$$

The above two Eqs. (60) and (61) explain the relations for density inhomogeneity, Weyl tensor, anisotropy and dissipation under the effects of gravitation effects related to exotic matter.

Equations (42) and (61) give

$$\left(X_{TF}^{(eff)} + 4\pi\mu^{(eff)}\right)' = -X_{TF}^{(eff)}\frac{3C}{C} + 4\pi q^{(eff)}B(\theta - \sigma), \quad (62)$$

The derived relation describes the density homogeneity through spatial derivative of effective density contribution. If $(\mu^{(eff)})' = (\mu^{(m)} + \mu^D)' = 0$, the density distribution is homogenous in the cluster. This implies that the homogenous evolution of cluster totally depends upon the behavior of baryonic as well as non-baryonic density.

Now we discuss this phenomenon in more details and find factors as well as condition for homogenous evolution. Firstly, we consider non dissipative case $(q^{(eff)}) = 0$, in this situation Eq.(62) implies

$$X_{TF}^{(eff)} = 0 \Leftrightarrow \mu'^{(eff)} = 0. \tag{63}$$

This shows that, in non dissipative case, homogenous evolution of the cluster is controlled by scalar $X_{TF}^{(eff)}$. This condition along with (32) implies

$$\varepsilon + \Pi^{(eff)} = 0, \tag{64}$$

which is possible if either $\varepsilon = 4\pi\Pi^{(eff)} = 0$, or $\varepsilon = -4\pi\Pi^{(eff)}$. The first case indicates that the evolution is also isotropic as well as conformally flat while the second one shows that effective anisotropic effects and tidal forces effects (describe by ε) are balancing each other.

If the evolving cluster of stars create dissipation then we have

$$X_{TF}^{(eff)} = 0 \Leftrightarrow \mu'^{(eff)} = q^{(eff)} B(\theta - \sigma) = q^{(eff)} B \frac{3\dot{R}}{R}. \tag{65}$$

This shows that homogenous evolution is controlled by effective dissipative along with scalar $X_{TF}^{(eff)} = 0$, expansion parameter and shear tensor. For expansion as well as shear free evolution the dissipation of the system will not played any role for evolution and the evolution has homogenous density along with $X_{TF}^{(eff)} = 0$.

From the above discussion, it can be noticed that non-dissipative homogenous evolution of cluster of stars containing non-baryonic matter (dark matter) depends upon two conditions, that is isotropic and conformally flat or anisotropic pressure balancing tidal forces. The dissipative homologous evolution depends upon dissipative factors due to matter as well as dark matter along with expansion and shear effects. Here the dissipation due to dark matter might be some sort of squandering of non-baryonic particles. The dissipative cluster having expansion and shear effects shows density inhomogeneity during evolution.

The Homologous Evolution

The homologous evolution means the evolving cluster of stars is preserving position, structure or origin. To discuss this case, we explore the behavior of evolving velocity U. From Eqs.(12), (48) and (22), we have

$$D_C\left(\frac{U}{C}\right)=-\frac{4\pi}{RE(1+\epsilon R}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}-\frac{2\dot{B}R'}{B}\right)2\epsilon+KABq^{(m)}\right]+\frac{\sigma}{C}, \tag{66}$$

after integrating we get

$$U=\tilde{a}(t)C+\int_0^r\left(\frac{\sigma}{C}-\frac{4\pi}{RE(1+\epsilon R)}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}-\frac{2\dot{B}R'}{B}\right)2\epsilon\right.\right.$$
$$\left.\left.+KABq^{m}\right]\right)C'dr, \tag{67}$$

where $\tilde{a}$ is an integrating function

$$U=\frac{U_\Sigma}{C_\Sigma}C-\int_r^{r_\Sigma}\left(\frac{\sigma}{C}-\frac{4\pi}{RE(1+\epsilon R)}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}-\frac{2\dot{B}R'}{B}\right)2\epsilon\right.\right.$$
$$\left.\left.+KABq^{(m)}\right]\right)C'dr. \quad (68)$$

The above equation governed the homologous evolution of the cluster. If the integral of (68) vanishes then we have

$$U=\tilde{a}(t)C, \quad \tilde{a}(t)\equiv\frac{U_\Sigma}{C_\Sigma}, \quad (69)$$

or

$$U\sim C \quad (70)$$

which is a feature of homologous evolution in Newtonian hydrodynamics. It can be observed from (68), if the cluster does not show shear as well as dissipative effects, then the velocity of evolving cluster is monitored by high curvature terms (dark matter terms) in the radial direction. Thus there are two possibilities for homologous evolution in non-dissipative situation, for shear-free case $(\sigma=0)$, the effects of high curvature terms vanishes $f(R)\sim R$. Whereas in shear case the dark matter terms canceled out the shear effects and the integral in Eq.(68) vanishes. It can noticed that, in GR, such type of evolution take place in shear-free and non-dissipative case but in higher curvature scenario the homologous evolution can take place in shear case as well.

The relativistic homologous condition is derived by considering two concenteric shells having areal radius C_1 and C_2 at $r = r_1 = constant$, and $r = r_2 = constant$, respectively, given by

$$\frac{C_1}{C_2} = constant. \tag{71}$$

The critical factor that can be noticed here is that condition (70) does not implies condition (71). By applying Eq.(70) for two shells of stellar fluid $1,\ 2$, the homologous condition turns out to be

$$\frac{U_1}{U_2} = \frac{A_1\mathcal{C}_1}{A_2\mathcal{C}_2} = \frac{C_1}{C_2}, \tag{72}$$

this implies (71) only if $A = A(t)$. By using simple coordinate transformation, we can convert $A = constant$ which is a property of geodesic fluid $(a = \frac{A'}{AB} = 0)$. This shows that for non-relativistic system the condition (71) satisfy whenever $U \sim C$, whereas for relativistic case the condition (70) implies (71), only if the evolving system is geodesic.

Thus, from Eq.(68) the quasi-homologous evolution take place if

$$\frac{-4\pi B}{RC'(1+\epsilon R)}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}-\frac{2\dot{B}R'}{B}\right)2\epsilon+KABq^{(m)}\right]+\frac{\sigma}{C}=0. \tag{73}$$

This implies that in the absences of shear and dissipation the high curvature terms associated to dark matter controlled the homologous evolution. In both cases dissipative or non-dissipative dark matter terms play a important role in controlling homologous condition.

It can be observed that according to the composition to cluster of stars, we cannot remove dark matter effects in the evolution of cluster that is

$$\frac{-4\pi B}{RC'(1+\epsilon R)}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}+\frac{2\dot{B}R'}{B}\right)2\epsilon\right]\neq 0. \tag{74}$$

So the stability of homologous condition occurs whenever

$$\frac{-4\pi B}{RC'(1+\epsilon R)}\left[\left(\dot{R}'-\frac{3A'\dot{R}}{A}-\frac{2\dot{B}R'}{B}\right)2\epsilon+KABq^{(m)}\right]=-\frac{\sigma}{C}. \tag{75}$$

In contrast to GR results, Eqs.(74) and (75) show that the homologous evolution of cluster of star can be dissipative and shear-free or can be non-dissipative with shear effects.

CONCLUSION

- We have studied cluster of stars in spherically symmetrical form where the fluid particles are a combination of baryonic (star clusters) and non-baryonic (dark matter). We have used the concept of high curvature gravity (f(R) gravity) to include dark terms in the discussion.

- The structure scalars are used for exploring dynamics of self-gravitating structures. These factors suggest refining the stellar evolution mechanism by reducing the system's complexity.

- It is found that these scalars controls aspects of Weyl tensor, effective anisotropic pres- sure and effective dissipative as well as inhomogeneous energy density factors in the evolution.

- We have studied evolution with homogenous density and isotropic case. It can be noticed that non-dissipative evolution shows density homogeneity if either the fluid is isotropic and conformally flat or anisotropic pressure of the fluid counteract the tidal forces effects.

CONCLUSION

- The dissipative case depends upon dissipative factors due to matter as well as dark matter along with expansion and shear scalars. The dissipating cluster having expansion and shear effects shows density inhomogeneity. For expansion and shear free situation, the dissipative evolution shows isotropic behavior with homogenous density.
- For non-relativistic homologous evolution, it has found that for if the cluster does not show shear as well as dissipative effects, then the velocity of evolving cluster is monitored by high curvature terms (dark matter terms) in the radial direction.
- Moreover, in contrast to GR results, it is concluded that the homologous evolution of cluster of star can be dissipative and shear-free or can be non-dissipative with shear effects.
- The geodesic evolution also have investigated and it is concluded that the star clusters moves with constant velocity. But around the center the uid becomes homologous which shows geodesic and homologous imply each other.

Thankyou