Holographic superconductors at zero density

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Outline

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- Holographic superconductors at zero density
- Our model
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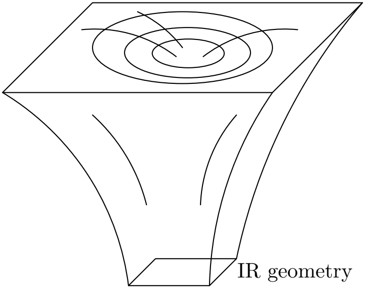
Motivation

Although holographic superconductors were intensely studied during the past 14 years, exact, analytic solutions of either the background geometry dual to the superconducting phase or the frequency-dependent conductivity are extremely rare.

IR geometries:

- $AdS_2 \times \mathbb{R}^2$
- Conformal to $AdS_2 \times \mathbb{R}^2$
- AdS_4
- Hyperscaling-violating geometry
- Timelike Kasner geometry
- $\ldots$



Analytic UV completion?

Zero density superconductors

- Metal: finite density system. Dual to a charged black hole
- Capacitor: zero density system. Conductivity is due to pair production. Dual to a neutral black hole

Zero density systems can be superconducting, and can be realized by the frustrated Hubbard model at half-filling on a square lattice

$$H = t \sum_{\langle i,j \rangle} c_i^\dagger c_j + t' \sum_{\langle\langle i,j \rangle\rangle} c_i^\dagger c_j + U \sum_i n_{i\uparrow} n_{i\downarrow} \,,$$

where t is the nearest neighbour hopping, t' is the next-nearest neighbour hopping, and U is the interaction. It was shown that superconductivity occurs in the phase diagram as a function of U/t and t'/t. It is desirable to construct a holographic dual to a superconductor at zero density.

[Nevidomskyy, Scheiber, Sénéchal, Tremblay, 0711.0214]

Highlights

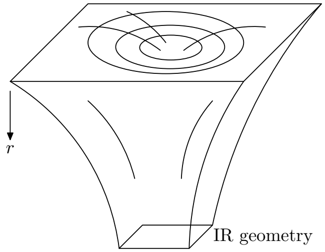
In our model of holographic superconductors at zero density, the ground state is described by

$$ds^2 = f\,(-dt^2 + d\vec{x}^2) + f^{-1} dr^2\,, \qquad A = 0\,,$$

$$f = \frac{r^2}{L^2}\left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\,, \qquad e^{\alpha\phi} = \left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\,.$$

Its IR geometry is hyperscaling-violating (HV) geometry

$$ds^2 = \tilde{r}^{\theta}\left(-\frac{dt^2}{\tilde{r}^{2z}} + \frac{d\tilde{r}^2 + dx^2 + dy^2}{\tilde{r}^2}\right)\,, \qquad z = 1\,, \quad \theta = \begin{cases} \frac{2}{1-\alpha^2} & (b > 0), \\ \frac{2\alpha^2}{\alpha^2 - 1} & (b < 0). \end{cases}$$



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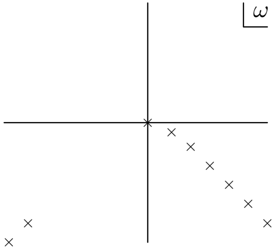
- This is (perhaps) the simplest model/solution that has a HV geometry in the IR and asymptotically AdS in the UV.
- This solution comes from a nontrivial neutral limit of an EMD system, and it helps us to have a better understanding of supergravity solutions.
- We obtain an analytic solution of the AC conductivity for a holographic superconductors from M-theory.

Introduction

- Holographic superconductors are AdS black holes that can spontaneously develop a hair.
- The “minimal” model of holographic superconductors has a charged black hole and a complex scalar field. The black hole has instability, and develops a scalar hair below a critical temperature.
- There are two types of instability. One is a zero mode, which is a pole of the retarded Green’s function at the frequency $\omega = 0$. The other is the IR instability, which happens when the exponent of the near-horizon AdS_2 geometry becomes imaginary.



$$\Delta_\pm = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$$

Introduction

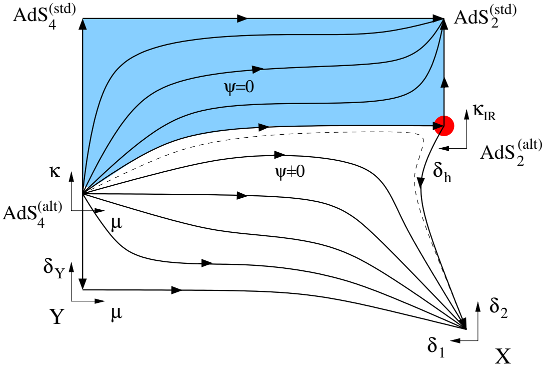
- A mechanism for holographic superconductors at zero density was proposed in [Faulkner, Horowitz & Roberts, 1008.1581].
- Consider the Schwarzschild-AdS (neutral) black hole and a scalar field.
- The black hole does not have an instability if we use either the standard or alternative quantization for the scalar.
- The black hole can develop an instability with a sourceless boundary condition, which corresponds to a double-trace deformation in the dual CFT.
- The superconducting instability is triggered by a zero mode when a double-trace deformation is present, and the neutral black hole develops a scalar hair below a critical temperature.

Introduction

Faulkner, Horowitz & Roberts, 1008.1581

$$\phi = \phi_a z^{\Delta_-}(1+\cdots) + \phi_b z^{\Delta_+}(1+\cdots),$$
$$S \to S - \kappa \int d^3x\, \mathcal{O}^2 \,.$$

The double-trace deformation corresponds to a new boundary condition on ϕ: $\phi_b = \kappa\phi_a$.



Introduction

Faulkner, Horowitz & Roberts, 1008.1581

Near the AdS boundary, the expansion of the scalar field is

$$\phi = \phi_a z + \phi_b z^2 + \cdots \tag{1}$$

To have a relevant deformation, we start from the alternative quantization, in which $\langle \mathcal{O} \rangle = \phi_a$. We modify the action by

$$S \to S - \kappa \int d^3x\, \mathcal{O}^2 \tag{2}$$

The Green's function for the scalar operator is

$$G^{(\kappa)} = \frac{1}{G^{-1} + \kappa} \tag{3}$$

A zero mode happens

$$\phi_b = \kappa \phi_a \,. \tag{4}$$

At a critical temperature T_c, the black hole is unstable against perturbations of the scalar field.

Introduction

We start with a general model of Stückelberg holographic superconductors, which have spontaneous breaking of the U(1) symmetry at low temperatures.

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{Z(\phi)}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\mathcal{W}(\phi)}{2}(\partial_\mu p - A_\mu)^2,$$

$$V = -\frac{6}{L^2}+\frac{1}{2}m^2\phi^2+\mathcal{O}(\phi^3), \quad Z = 1+g_Z\phi^2+\mathcal{O}(\phi^3), \quad \mathcal{W} = g_{\mathcal{W}}\phi^2+\mathcal{O}(\phi^3)\,.$$

The gauge symmetry is $A_\mu \to A_\mu + \partial_\mu\alpha$ and $p \to p + \alpha$.
[Franco, García-García & Rodríguez-Gómez, 0906.1214]

Comparison: The minimal model with a complex scalar field $\psi = \phi e^{ip}$ is

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\nabla\psi - iqA\psi|^2 - V(|\psi|)\,,$$

In the more general model, we can treat ϕ and p as any real field, and $V(\phi)$, $Z(\phi)$, and $\mathcal{W}(\phi)$ are not necessarily even functions of ϕ.

Introduction

We start with a general model of Stückelberg holographic superconductors, which have spontaneous breaking of the U(1) symmetry at low temperatures.

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$$V = -\frac{6}{L^2}+\frac{1}{2}m^2\phi^2+\mathcal{O}(\phi^3), \quad Z = 1+g_Z\phi^2+\mathcal{O}(\phi^3), \quad \mathcal{W} = g_{\mathcal{W}}\phi^2+\mathcal{O}(\phi^3)\,.$$

The gauge symmetry is $A_\mu \to A_\mu + \partial_\mu\alpha$ and $p \to p + \alpha$. We choose the gauge $p = 0$. We are interested in neutral black hole solutions to the system. When $A_\mu = 0$, the above system shares the same background geometry as an Einstein-scalar system

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\,.$$

In other words, an Einstein-scalar system can be promoted to a holographic superconductor at zero density, if the scalar hair is spontaneously developed.

Motivation for the scalar potential

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\,.$$

$$V(\phi) = -\frac{2}{(1+\alpha^2)^2L^2}\left[\alpha^2(3\alpha^2-1)e^{-\phi/\alpha} + 8\alpha^2 e^{(\alpha-1/\alpha)\phi/2} + (3-\alpha^2)e^{\alpha\phi}\right].$$

- The values of $\alpha = 0$, $1/\sqrt{3}$, 1, and $\sqrt{3}$ correspond to special cases of STU supergravity. Those are the simplest and most reliable solutions in the applications of AdS/CFT.
- This potential was derived by different people in several different ways. [Gao & Zhang, hep-th/0411104]
- The planar, spherical, and hyperbolic black hole solutions are "simply related". [JR, 1910.06344]
- The ground state solution is obtained by taking a nontrivial neutral limit of an EMD system whose special cases are STU supergravity.

STU supergravity

The AdS_4 Lagrangian is

$$\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 + 8g^2(\cosh\phi_1 + \cosh\phi_2 + \cosh\phi_3) - \frac{1}{4}\sum_{i=1}^{4} e^{\vec{a}_i\cdot\vec{\phi}}(F^i_{(2)})^2,$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, $\vec{a}_1 = (1,1,1)$, $\vec{a}_2 = (1,-1,-1)$, $\vec{a}_3 = (-1,1,-1)$, and $\vec{a}_4 = (-1,-1,1)$. The solution is given by

$$ds^2 = -(H_1H_2H_3H_4)^{-1/2}fdt^2 + (H_1H_2H_3)^{1/2}(f^{-1}dr^2 + r^2d\Sigma^2_{2,k}),$$

$$X_i = H_i^{-1}(H_1H_2H_3H_4)^{1/4}, \qquad A^i_{(1)} = \sqrt{k}(1 - H_i^{-1})\coth\beta_i dt$$

where

$$f = k - \frac{\mu}{r} + \frac{4}{L^2}r^2(H_1H_2H_3H_4), \qquad H_i = 1 + \frac{\mu\sinh^2\beta_i}{kr}$$

If we set the chemical potential be zero, the dilaton fields will also be zero.

Gubser-Rocha model

The 3-charge black hole in AdS_4 is determined by

$$\mathcal{L} = R - \frac{1}{4}e^{-\frac{1}{\sqrt{3}}\phi}F^2 - \frac{1}{2}(\partial\phi)^2 + \frac{6}{L^2}\cosh\frac{\phi}{\sqrt{3}}\,. \tag{5}$$

In the ordinary form, the solution is

$$ds^2 = e^{2\mathcal{A}}(-hdt^2 + d\vec{x}^2) + \frac{e^{2\mathcal{B}}}{h}dr^2, \tag{6}$$

$$\mathcal{A} = \ln\frac{\bar{r}}{L} + \frac{3}{4}\ln\left(1 + \frac{Q}{\bar{r}}\right), \qquad \mathcal{B} = -\mathcal{A}, \qquad h = 1 - \frac{(\bar{r}_h + Q)^3}{(\bar{r} + Q)^3}, \tag{7}$$

$$A = \frac{\sqrt{3Q(\bar{r}_h + Q)}}{L}\left(1 - \frac{\bar{r}_h + Q}{\bar{r} + Q}\right)dt, \qquad \phi = -\frac{\sqrt{3}}{2}\ln\left(1 + \frac{Q}{\bar{r}}\right), \tag{8}$$

Apparently, the neutral limit of this black hole is at $Q = 0$, which is nothing but the planar Schwarzschild-AdS black hole. However, a closer examination reveals that this is not the whole story. A different parametrization gives a metric with a nontrivial dilaton profile.

Neutral limits of EMD

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}e^{-\alpha\phi}\cancel{F}^2 - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right),$$

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + U(r)d\Sigma_{2,k}^2,$$

$$\cancel{A} = 2\sqrt{\frac{b\cancel{c}}{1+\alpha^2}}\left(\frac{1}{r_h} - \frac{1}{r}\right)dt, \qquad e^{\alpha\phi} = \left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}},$$

$$f = \left(k - \frac{\cancel{c}}{r}\right)\left(1 - \frac{b}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + \frac{r^2}{L^2}\left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \qquad U = r^2\left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}.$$

[Gao & Zhang, hep-th/0411104]

- $b = 0$. The black hole becomes neutral, and the scalar field is zero.
- $c = 0$. A neutral solution is obtained, and the scalar field is nontrivial. [JR, 1910.06344]
    - $k = 0, 1$: This is not a black hole.
    - $k = -1$: We obtain a hyperbolic black hole, which is different from the hyperbolic Schwarzschild-AdS black hole.

Neutral limits of an EMD system

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + U(r)d\vec{x}^2,$$

$$\cancel{A} = 2\sqrt{\frac{b\cancel{c}}{1+\alpha^2}}\left(\frac{1}{r_h} - \frac{1}{r}\right)dt, \qquad e^{\alpha\phi} = \left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}},$$

$$f = \left(0 - \frac{\cancel{c}}{r}\right)\left(1 - \frac{b}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + \frac{r^2}{L^2}\left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \qquad U = r^2\left(1 - \frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}.$$

- $b = 0$. The black hole becomes neutral, and the scalar field is zero.
- $c = 0$. A neutral solution is obtained, and the scalar field is nontrivial. We treat this solution as the ground sate of holographic superconductors at zero density.

To support this claim, we need to find finite temperature solutions under the following conditions: (1) The boundary condition for the scalar is sourceless. (2) The hairy solution has lower free energy below a critical temperature. (3) When we take the extremal limit, the finite temperature solution approaches to the ground state solution.

Gubser criterion

The Gubser criterion [hep-th/0002160] is a way to justify a spacetime singularity in the IR.

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\,.$$

$$V(\phi) = -\frac{2}{(1+\alpha^2)^2L^2}\left[\alpha^2(3\alpha^2-1)e^{-\phi/\alpha} + 8\alpha^2 e^{(\alpha-1/\alpha)\phi/2} + (3-\alpha^2)e^{\alpha\phi}\right].$$

$$ds^2 = f(-dt^2 + d\vec{x}^2) + f^{-1}dr^2, \qquad A = 0\,,$$

$$f = \frac{r^2}{L^2}\left(1-\frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \qquad e^{\alpha\phi} = \left(1-\frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}.$$

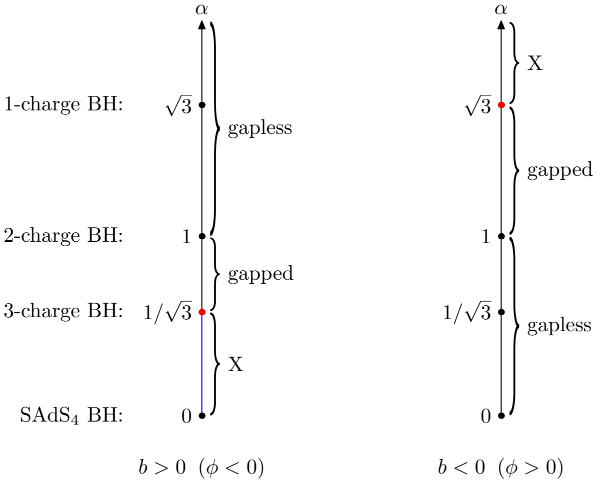
There are two statements of the Gubser criterion. It is easy to check they agree for our model.

- The scalar potential is bounded from above in the solution.
- The geometry can be obtained as the extremal limit of a regular black hole.

The Gubser criterion is stronger than the null energy condition (NEC).

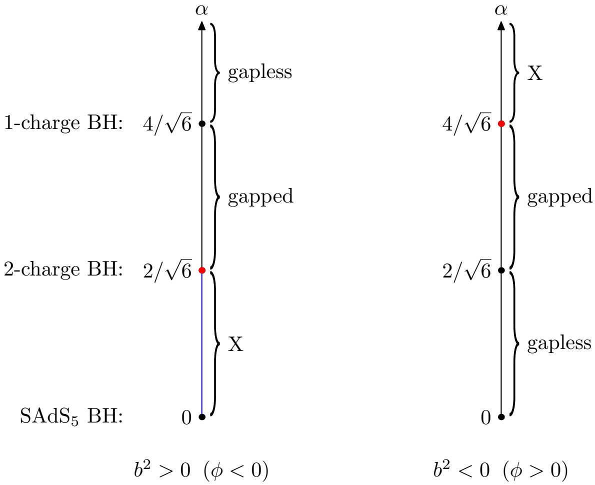
Classification by IR geometries: AdS_4

$$V(\phi) = -\frac{2}{(1+\alpha^2)^2 L^2}\left[\alpha^2(3\alpha^2-1)e^{-\phi/\alpha} + 8\alpha^2 e^{(\alpha-1/\alpha)\phi/2} + (3-\alpha^2)e^{\alpha\phi}\right].$$



Classification by IR geometries: AdS_5

$$V(\phi) = -\frac{12}{(4+3\alpha^2)^2L^2}\left[3\alpha^2(3\alpha^2-2)e^{-\frac{4\phi}{3\alpha}}+36\alpha^2 e^{\frac{3\alpha^2-4}{6\alpha}\phi}+2(8-3\alpha^2)e^{\alpha\phi}\right],$$



Superconducting phase

We obtain the finite temperature solution to the Einstein-scalar system.

$$ds^2 = \frac{1}{z^2}\left(-g(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{g(z)} + dx^2 + dy^2\right). \quad (9)$$

Near the AdS boundary $z = 0$, the asymptotic behavior of the functions is

$$g = 1 + \frac{1}{4}\phi_a^2 z^2 + g_3 z^3 + \cdots , \quad (10)$$

$$e^{-\chi} g = 1 - \frac{1}{2} m_0 z^3 + \cdots \quad (11)$$

$$\phi = \phi_a z + \phi_b z^2 + \cdots , \quad (12)$$

The boundary condition is

$$\phi_b = \kappa \phi_a . \quad (13)$$

Holographic renormalization

The free energy is calculated by the renormalized on-shell action as $F/T = S_E + S_{\text{ct}}$, where S_E is the Euclidean action and S_{ct} is boundary counter terms. The Euclidean on-shell action is a total derivative

$$S_E = \int d^3x \int_1^0 dz \left(\frac{2}{z^3} g e^{-\chi/2} \right)' = \int d^3x \, \frac{2}{z^3} g e^{-\chi/2} \bigg|_{z=0} .$$

The boundary terms are

$$S_{\text{ct}} = \int d^3x \sqrt{-\gamma} \left(-2K + \frac{4}{L} + \phi n^\mu \partial_\mu \phi - \frac{1}{2L}\phi^2 - L^2(\phi_a \phi_b - W(\phi_a)) \right) \bigg|_{z=0} ,$$

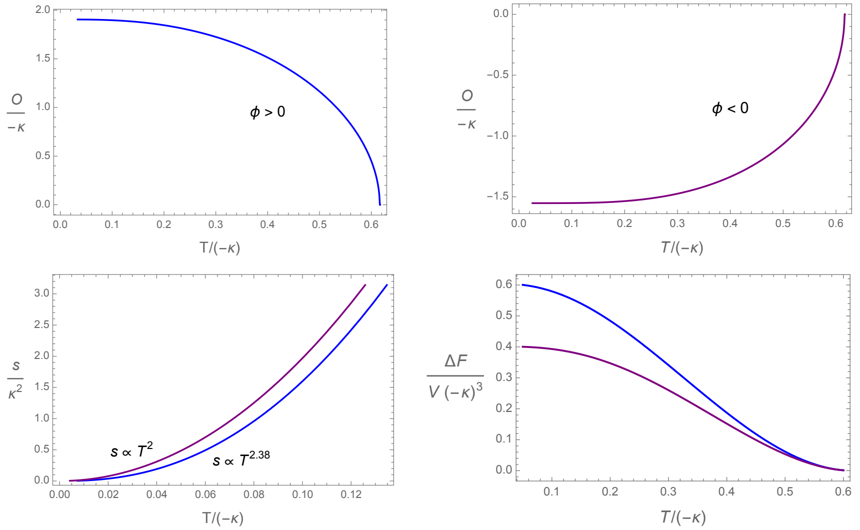
for the general boundary condition $\phi_b = W'(\phi_a)$. The finite terms are chosen such that the following thermodynamic law is satisfied:

$$\delta \text{f} = -s\delta T - (\phi_b - W'(\phi_a))\delta\phi_a .$$

For the double-trace deformation, we have $W(\phi_a) = (1/2)\kappa\phi_a^2$. The free energy density is give by

$$\text{f} = -\frac{1}{2} m_0 + \frac{1}{6}\phi_a\phi_b = g_3 - \frac{1}{2}\phi_a\phi_b .$$

Superconducting phase



Superconducting phase

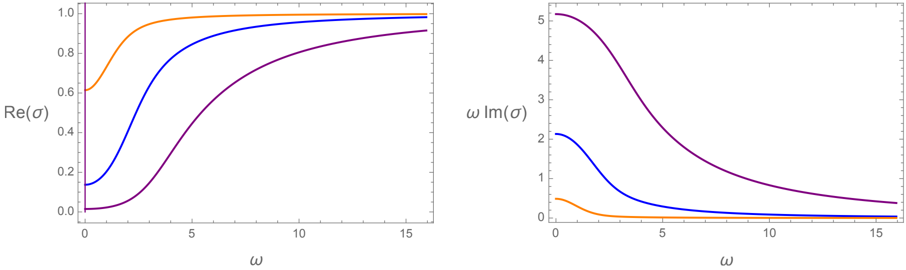


Figure: The behavior of Re(σ) and ωIm(σ) as a function of frequency. It approaches to a constant as $\omega \to 0$, which implies that there are a pole in Im(σ) and a delta function in Re(σ) at $\omega = 0$. The orange, blue, and purple curves correspond to $T/T_c = 0.5$, 0.2, 0.1 ($\kappa = -0.8$, -2.2, -5), respectively.

A holographic superconductor from M-theory

We find that the following model of holographic superconductors from M-theory studied in [Donos & Gauntlett 1104.4478; Aprile 1206.5827] admit an analytic solution of the AC conductivity at zero density:

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 + \frac{2}{L^2}(\cosh\phi + 2) - \frac{1}{4}F^2 - \frac{1}{L^2}\sinh^2\left(\frac{\phi}{2}\right)A^2, \qquad (14)$$

At zero density, the background geometry for the ground state is

$$ds^2 = \frac{r(r-b)}{L^2}(-dt^2 + d\vec{x}^2) + \frac{dr^2}{r(r-b)}. \qquad (15)$$

The real part of the conductivity is

$$\text{Re}[\sigma(\omega)] = \frac{\pi|b|}{2L^2}\delta(\omega) + \theta(4\omega^2L^4 - b^2)\frac{\sqrt{4\omega^2L^4 - b^2}}{2\omega L^2}, \qquad (16)$$

where $\theta(x)$ is the Heaviside step function. There are a delta function at $\omega = 0$, and a gap in $0 < 2\omega L^2 < b$.

Summary

- The ground state of the superconductor is analytically available. The geometry has a HV geometry in the IR and asymptotically AdS in the UV.
- This solution comes from a nontrivial neutral limit of an EMD system, and it helps us to have a better understanding of supergravity solutions.
- We numerically construct the finite temperature solution and show that there are phase transitions. We also calculate the AC conductivity, and verify that there is a superconducting delta function, and a sum rule is satisfied as the temperature varies.
- We obtain an analytic solution of the AC conductivity for a holographic superconductors from M-theory.