Cosmological gravitational wave damping as estimated by linearized perturbations on null cone coordinates

Petrus van der Walt, Nigel Bishop & Monos Naidoo

Department of Mathematics
Rhodes University
Grahamstown, South Africa

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RELATED REFERENCES

N.T. Bishop, M. Naidoo, and P.J. van der Walt, *Effect of a viscous fluid shell on the propagation of gravitational waves* arxiv (2022)
M. Naidoo, N.T. Bishop, and P.J. van der Walt, *Modifications to the signal from a gravitational wave event due to a surrounding shell of matter* Gen.Rel. Grav. **53** 77 (2021)
N.T. Bishop, M. Naidoo, and P.J. van der Walt, *Effect of a low density dust shell on the propagation of gravitational waves* Gen.Rel. Grav. **52** 92 (2020)

These originate from a paper published in 2005:

N.T. Bishop, *Linearized solutions of the Einstein equations within a Bondi–Sachs framework, and implications for boundary conditions in numerical simulations* Class. Quantum Grav. **22** 2393 (2005)

OVERVIEW

- Mathematical development: Background solution Bondi-Sachs formalism
- Mathematical development: Linearized perturbed solution
- Mathematical development: Thin dust shell approximation
- Mathematical development: Energy loss due to viscosity
- Astrophysical applications: CCSNe
- Cosmological application: Primordial GWs
- Cosmological conclusion

BACKGROUND SOLUTION: BONDI-SACHS FORMALISM

The Bondi-Sachs metric is

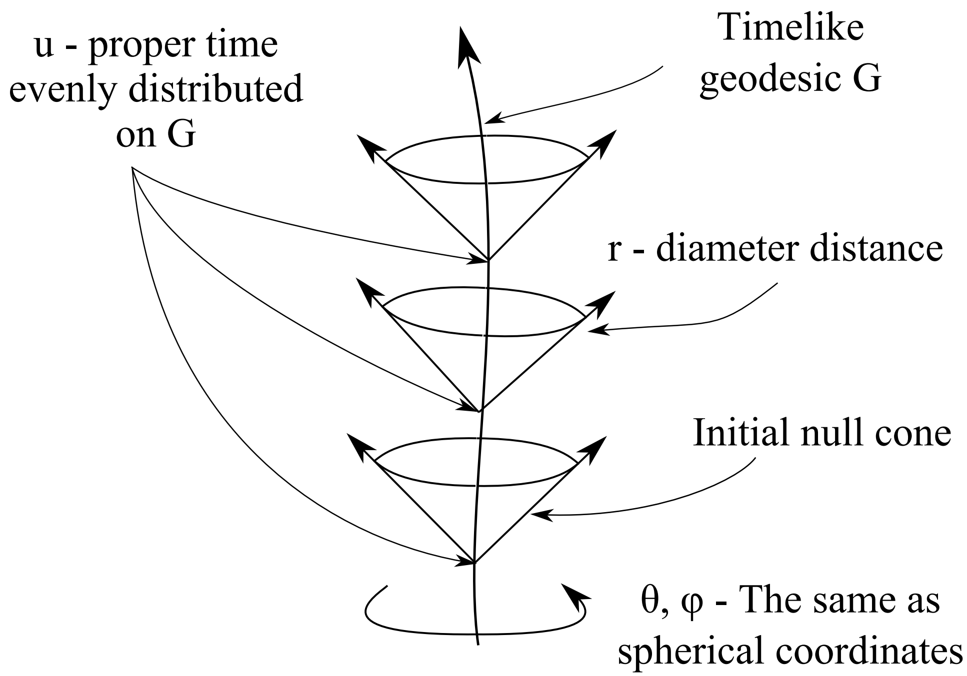
$$ds^2 = -\left(e^{2\beta}\left(1 + W_c r\right) - r^2 h_{AB} U^A U^B\right) du^2$$
$$-2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B,$$

where r is an area coordinate so that $det(h_{AB}) = det(q_{AB})$ with q_{AB} a unit sphere metric (e.g. $d\theta^2 + \sin^2\theta d\phi^2$). In Minkowski spacetime $\beta = U^A = 0$, $h_{AB} = q_{AB}$, $W_c = 0$. We introduce a complex dyad q_A (e.g. $q_A = (1, i\sin\theta)$). Then h_{AB}, U^A can be represented by

$$J = h_{AB} q^A q^B / 2\,, \quad U = U^A q_A\,,$$

with $J \neq 0$ characterizing a deviation from spherical symmetry. In an appropriate gauge, J is directly related to the polarization states of a gravitational wave, $J = h_+ + i h_\times$.

BACKGROUND SOLUTION: BONDI-SACHS FORMALISM



SOLUTION WITH LINEARIZED PERTURBATIONS

We make the ansatz of a small perturbation about Minkowski spacetime

$$
\begin{aligned}
\beta =& \Re(\beta^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, \quad U = \Re(U^{[2,2]}(r)e^{i\nu u})_1 Z_{2,2}\,, \\
W_c =& \Re(W_c^{[2,2]}(r)e^{i\nu u})_0 Z_{2,2}\,, \quad J = \Re(J^{[2,2]}(r)e^{i\nu u})_2 Z_{2,2}\,,
\end{aligned}
$$

The perturbations oscillate in time with frequency $\nu/(2\pi)$. The quantities ${}_sZ_{\ell,m}$ are "real" spin-weighted spherical harmonic basis functions related to the usual ${}_sY_{\ell,m}$. We consider a source that is continuously emitting GWs at constant frequency dominated by the $\ell = 2$ (quadrupolar) components.

SOLUTION WITH LINEARIZED PERTURBATIONS

Solving the vacuum Einstein equations with no incoming radiation leads to

$$\begin{aligned}
\beta^{[2,2]} =& b_0 \,, \\
W_c^{[2,2]} =& 4i\nu b_0 - 2\nu^4 C_{40} - 2\nu^2 C_{30} + \frac{4i\nu C_{30} - 2b_0 + 4i\nu^3 C_{40}}{r} \\
&+ \frac{12\nu^2 C_{40}}{r^2} - \frac{12i\nu C_{40}}{r^3} - \frac{6C_{40}}{r^4} \,, \\
U^{[2,2]} =& \frac{\sqrt{6}(-2i\nu b_0 + \nu^4 C_{40} + \nu^2 C_{30})}{3} + \frac{2\sqrt{6} b_0}{r} + \frac{2\sqrt{6} C_{30}}{r^2} \\
&- \frac{4i\nu\sqrt{6} C_{40}}{r^4} - \frac{3\sqrt{6} C_{40}}{r^4} \,, \\
J^{[2,2]} =& \frac{2\sqrt{6}(2b_0 + i\nu^3 C_{40} + i\nu C_{30})}{3} + \frac{2\sqrt{6} C_{30}}{r} + \frac{2\sqrt{6} C_{40}}{r^3} \,.
\end{aligned}$$

Defining the rescaled GW strain by $\mathcal{H}_0 = r(h_+ + i h_\times)$, we find

$$\mathcal{H}_0 = \Re(H_0 \exp(i\nu u))\, {}_2Z_{2,2} \text{ with } \; H_0 = -2\sqrt{6}\nu^2 C_{40} \,.$$

C_{40} is determined by the physics, and b_0, C_{30} are gauge freedoms.

RESULT FOR A DUST SHELL

We consider the physical problem of a spacetime that is empty except in a shell located at $r_0 < r < r_0 + \Delta$. where there is a spherically symmetric distribution of dust with a density profile that vanishes at $r = r_0$ and $r = r_0 + \Delta$.

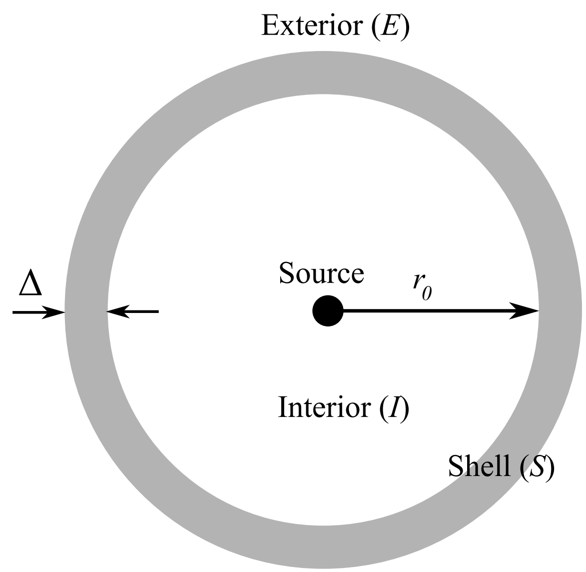


Figure 1: Thin shell

RESULT FOR A DUST SHELL

Let M_S be the shell mass, then the GW is

$$H = -2\nu^2\sqrt{6}C_{40}\left(1 + \frac{2M_S}{r_0} - \frac{2iM_S}{r_0^2\nu} + \frac{iM_S e^{-2ir_0\nu}}{2r_0^2\nu} + \mathcal{O}\left(\frac{M_S\delta}{r_0^2}, \frac{M_S}{r_0^3\nu^2}\right)\right) .$$

- Term 1. Effect of matter shell neglected.
- Term 2. Effect of gravitational red-shift.
- Term 3. This is out of phase with the leading terms, and thus to $\mathcal{O}(M_S)$, does not affect the magnitude of H.
- Term 4. This affects the magnitude of H, and is due to an incomimg wave modifying the geometry near the source and thus the inspiral rate.

ENERGY LOSS DUE TO VISCOSITY

We use the formula that the rate of energy loss per unit volume is $-2\eta\sigma_{ab}\sigma^{ab}$. This quantity is evaluated, then integrated over a shell of radius r and thickness δr; the integration is straightforward because of the orthonormality of the angular basis functions. We find that the average rate of energy loss to the shell is

$$\left\langle \dot{E}_\eta \right\rangle = -12\eta C_{40}^2 \nu^6 \delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right) ,$$

where $\langle f \rangle$ denotes the average of $f(u)$ over a wave period, i.e.

$$\langle f \rangle = \frac{\nu}{2\pi} \int_0^{\frac{2\pi}{\nu}} f\, dt \,,$$

and where we have used $\left\langle \cos^2(\nu u) \right\rangle = \left\langle \sin^2(\nu u) \right\rangle = 1/2$ and $\langle \cos(\nu u) \sin(\nu u) \rangle = 0$.
The rate of energy output as GWs is

$$\left\langle \dot{E}_{GW} \right\rangle = \frac{3C_{40}^2 \nu^6}{4\pi} \,,$$

so that

$$\left\langle \dot{E}_\eta \right\rangle = -16\pi\eta\delta r \left\langle \dot{E}_{GW} \right\rangle \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8}\right) .$$

ENERGY LOSS DUE TO VISCOSITY

Energy conservation means that energy absorbed by the viscous fluid is balanced by a reduction in the GW energy. Thus

$$\left\langle \dot{E}_{GW} \right\rangle (r + \delta r) = \left\langle \dot{E}_{GW} \right\rangle (r) \times \left[1 - 16\pi\eta\delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8} \right) \right] .$$

H represents the magnitude of the GWs, and $\left\langle \dot{E}_{GW} \right\rangle \propto H^2$ so

$$H(r + \delta r) = H(r) \left[1 - 8\pi\eta\delta r \left(1 + \frac{2}{r^2\nu^2} + \frac{9}{r^4\nu^4} + \frac{45}{r^6\nu^6} + \frac{315}{r^8\nu^8} \right) \right] .$$

The resulting differential equation is solved to give

$$H(r) = C \exp \left(-8\pi\eta \left(r - \frac{2}{r\nu^2} - \frac{3}{r^3\nu^4} - \frac{9}{r^5\nu^6} - \frac{45}{r^7\nu^8} \right) \right) ,$$

where C is a constant.

ENERGY LOSS DUE TO VISCOSITY: TWO SPECIAL CASES

There are two useful special cases. Let r_i, r_o be the inner and outer radii of the shell. If r_i, r_o are much larger than the wavelength λ of the GWs, then

$$H(r_o) = H(r_i) \exp\left(-8\pi\eta(r_o - r_i)\right) \ . \tag{1}$$

Equivalent results have been given before[1].

If r_i is much smaller than the wavelength of the GWs with $r_o \gg r_i$, then

$$H(r_o) = H(r_i) \exp\left(-\frac{360\pi\eta}{r_i^7 \nu^8}\right) = H(r_i) \exp\left(-\frac{45\eta\lambda^8}{32 r_i^7 \pi^7}\right) \ . \tag{2}$$

To our knowledge, viscous damping of GWs with $r_i \ll \lambda$ has not been studied previously, and the result is new.

[1] E.g., Hawking S. W., Perturbations of an expanding universe, Astrophys. J. **145**, 544 (1966).

ASTROPHYSICAL APPLICATIONS

Details presented in the talk: Session C3, Wednesday July 6, 10:45-11:00, *Damping of gravitational waves originating from core-collapse supernovae*, by Monos Naidoo.

From:

$$H(r_o) = H(r_i)\exp\left(-\frac{360\pi\eta}{r_i^7\nu^8}\right) = H(r_i)\exp\left(-\frac{45\eta\lambda^8}{32r_i^7\pi^7}\right) ,$$

it follows that:

- The formulas above are in geometric units, and are converted to SI units on multiplication η by $G/c^3 = 2.477\times10^{-36}$s/kg. Reported values for η may be up to 10^{25}kg/m/s.
- A numerical relativity simulation with GW extraction far from the source would include the above effects. However, in situations such as CCSNe, GW extraction is calculated using a modified quadrupole formula.
- CCSNe includes processes with GW emission at 100Hz, so λ =3,000km, and $r_i \approx$ 15km. Thus $(\lambda/r_i)^7$ can be large, and GW damping may occur.

ASTROPHYSICAL CONCLUSIONS

- Changes to both phase and magnitude occur when GWs pass through a shell of matter, and these changes are large enough to be measurable in a GW signal from a close-by source.
- The effect of viscosity on GW propagation was calculated using energy considerations, and the known result for a viscous shell was recovered when $\lambda \ll r_i$.
- However, when $\lambda \gg r_i$, it was found that viscous damping of GWs can be a significant effect.

COSMOLOGICAL SCENARIO: PRIMORDIAL COSMOLOGY

Can we relate these results to cosmology? In the early Universe, we can consider viscous damping of pGWs in the primordial plasma.

- The effect of viscosity on pGWs originating from a discrete source in a primordial plasma.
- A thin region of high viscosity.
- A very specific inflation scenario.

However,

- Epochs in the early Universe have different matter phases and properties.
- pGWs produce fundamentally stochastic background(s).
- Quark-gluon plasma, viscosity can be negligible.

LET'S MODEL A HIGH VISCOSITY SCENARIO

At the present time, we have $t = 13.7$Gyr $(= 4.32 \times 10^{18} s)$, $T = 2.725$K and we normalise the *scale factor* to be $a_0 = 1$ at present.

Density of radiation and matter were approximately equal at $t = 56,000$yr $(1.7662 \times 10^{12}$s) and $T = 9,000K$

Consider an inflation scenario where the at end of inflation at $t = 10^{-32}s$, $T = 10^{27}$ to $10^{28}K$, $1/H = a/\dot{a} = 10^{-35}s$; thus *horizon scale* is $c/H = 3 \times 10^{-27}m$.

According to Misner, Thorne & Wheeler [2], eq, (28.1), T is proportional to $1/a$ Thus, at the end of inflation, $a = 10^{-27}$.

Now, the wavelength is proportional to a. GWs detectable now have frequency 10^{-9} to 10^{3}Hz, so wavelengths are in the range 3×10^{5}m to 3×10^{17}m. Thus at end of inflation wavelength is in the range 3×10^{-22} to 3×10^{-10}m.

[2]Misner, C. W., Thorne, K. S. & Wheeler, J. A., Gravitation, W.H. Freeman (1973).

LET'S MODEL A HIGH VISCOSITY SCENARIO

From Weinberg 1971 [3] Eq. (3.20) rewritten to make the units consistent:

$$\tau = (16\pi G\eta/c^2)^{-1}$$

Eq. (3.21) rewritten to make the units consistent:

$$\eta = \frac{4}{15} aT^4\tau/c$$

Eliminating τ from the two equations, we get

$$\eta^2 = \frac{aT^4c}{60\pi G}$$

When $T = 10^{27}$K,

$$\eta = 3.68 \times 10^{58} kg/m/s$$

This value of η is very large, and even when multiplied by G/c^3, we get a value of 9.09×10^{22}

Using,

$$H(r_o) = H(r_i)\exp\left(-8\pi\eta(r_o - r_i)\right) .$$

with $r\nu \gg 1$ and rewritten in terms of t, we have

$$\frac{dH}{dr} = -8\pi\eta H \rightarrow \frac{dH}{dt} = -8\pi\eta c H$$

[3] Weinberg S., Entropy Generation and the Survival of Protogalaxies in an Expanding Universe, Astrophys. J. **168**, 175 (1971).

LET'S MODEL A HIGH VISCOSITY SCENARIO

Now, η behaves as T^2, i.e. as $1/a^2$. Further, we are in the radiation dominated era, and a behaves as $t^{1/2}$. Thus η behaves as $1/t$ and we have $dH/dt = -8\pi\eta_i cHt_i/t$ with $\eta_i = 9.09 \times 10^{22}$ and $t_i = 10^{-33}$ Let $A = 8\pi\eta_i ct_i = 0.69$ Then integrating

$$\frac{dH}{H} = -A\frac{dt}{t} \quad \text{we get} \quad H_o = H_i\left(\frac{t_i}{t_o}\right)^{-A}$$

Some example values:

- $t_o = 10\ t_i$, $H_o = 0.2\, H_i$
- $t_o = 100\ t_i$, $H_o = 0.04\, H_i$
- If $T_i = 10^{28}$K, then $t_o = 10\ t_i$, $H_o = 10^{-69}\, H_i$

Otherwise, considering that the wavelength is in the rage in 3×10^{-22} to 3×10^{-10}m while the horizon scale is $c/H = 3 \times 10^{-27}m$, can can be used to justify the case for $r_i \ll \lambda$, then

$$H(r_o) = H(r_i)\exp\left(-\frac{360\pi\eta}{r_i^7\nu^8}\right) = H(r_i)\exp\left(-\frac{45\eta\lambda^8}{32r_i^7\pi^7}\right),$$

in which case (λ/r_i) can be very large. Using the same formulation, significant damping can then occur for even small values of η_i.

COSMOLOGICAL CONCLUSIONS

Current status:

- If r_i, r_o are much larger than the wavelength λ of the GWs, then

$$H(r_o) = H(r_i)\exp\left(-8\pi\eta(r_o - r_i)\right) .$$

- The values of T, t_i and η_i have significant effects on pGW damping, especially with high values of η_i.
- If r_i is much smaller than the wavelength of the GWs with $r_o \gg r_i$, then

$$H(r_o) = H(r_i)\exp\left(-\frac{360\pi\eta}{r_i^7\nu^8}\right) = H(r_i)\exp\left(-\frac{45\eta\lambda^8}{32r_i^7\pi^7}\right) .$$

- The damping is also dependent on λ.
- Depending on the ratio of (λ/r_i), the damping can be significant even for moderate values of η.

COSMOLOGICAL CONCLUSIONS

- Since r_i and r_o are related to t_i and t_o, the shell can be interpreted as a later epoch. For instance, the duration of a low interacting epoch might be represented by r_o when followed by an epoch where significant viscosity is expected.
- **These results can be useful for either incorporating damping effects in pGW detections or otherwise verifying physical expected properties (e.g. viscosity) from pGW detection or the lack thereof.**

Future developments:

- Extend the model to evaluate a stochastic background.
- Vary the viscosity with time evolution.
- Consider physical properties related to the various epochs.
- Relate results to observations.

ACKNOWLEDGEMENTS

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THANK YOU