Quantum Belinski-Khalatnikov-Lifshitz scenario

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Based on:

[1] P. Goldstein and W.P.,
"Generic instability of the dynamics underlying the Belinski-Khalatnikov-Lifshitz scenario",
Eur. Phys. J. C (2022) 82: 216.

[2] A. Góźdź, A. Pędrak, and W.P.,
"Quantum dynamics corresponding to the classical BKL scenario",
arXiv:2204.11274 [gr-qc].

Introduction

- Friedmann model (1922)
  - assumes isotropy and homogeneity of space
  - solution includes gravitational singularity
  - commonly used in astrophysics and cosmology
- Lifshitz analysis (1946): isotropy is unstable in the evolution towards singularity[1].
- In late 50-ties relativists (USSR, USA) began examination of models with homogeneous space (Bianchi-type models).

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Belinskii-Khalatnikov-Lifshitz (BKL) conjecture

- Dynamics of BVIII and BIX was analyzed[2] to get insight into the dynamics of spacetime near the cosmological spacelike singularity.
- BKL conjecture[3]:
  general relativity implies existence of generic solution that is singular
  - corresponds to non-zero measure subset of all initial data
  - is stable against perturbation of initial data
  - depends on proper number of arbitrary functions of space

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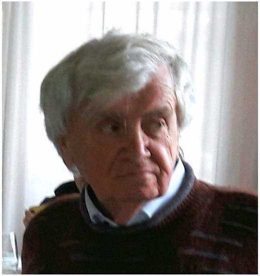

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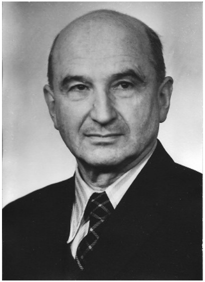

Evgeny Mikhailovich Lifshitz

BKL conjecture and singularity theorems

- The Penrose-Hawking singularity theorems (of 60-ties) mainly concern possible existence of incomplete geodesics in spacetime.
- Those theorems say little about the dynamics of gravitational field near singularities so that are of little usefulness in the context of finding corresponding quantum dynamics.
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Dynamics underlying BKL scenario

The asymptotic form (near the singularity) of the dynamical equations of general Bianchi VIII and IX models reads[4]:

$$\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}, \tag{1}$$

where $a = a(t),\ b = b(t),\ c = c(t)$ are directional scale factors, and t is a function of proper time.

The solutions to (1) must satisfy the constraint:

$$\frac{d \ln a}{dt}\frac{d \ln b}{dt} + \frac{d \ln a}{dt}\frac{d \ln c}{dt} + \frac{d \ln b}{dt}\frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{2}$$

Present the essence of the dynamics underlying the BKL scenario[5].

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BKL scenario

Remark!

- BKL scenario results from considering dynamics of BVIII and BIX which include some contributions from matter fields (corresponding metrics are nondiagonal);
- dynamics of the mixmaster universe[6] concerns vacuum BIX (corresponding metric is diagonal);
- BKL dynamics is different from mixmaster dynamics[7]

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Solution to BKL scenario

Quite recently, we have found analytical solution to the dynamics (1)–(2). It reads[8]:

$$\tilde{a}(t) = \frac{3}{t - t_0}, \quad \tilde{b}(t) = \frac{30}{(t - t_0)^3}, \quad \tilde{c}(t) = \frac{120}{(t - t_0)^5}, \tag{3}$$

where $t > t_0$ and where t_0 is an arbitrary real number.
This is the only solution in the Painlevé sense.
The solution (3) is unstable against small perturbation:

$$a(t) = \tilde{a}(t) + \epsilon\alpha(t), \tag{4a}$$

$$b(t) = \tilde{b}(t) + \epsilon\beta(t), \tag{4b}$$

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Solution to BKL scenario (cont)

Inserting (4) into (1)–(2) leads, in the first order in ϵ, to the following solution of the resulting equations:

$$\alpha(t) = \exp(-\theta/2)\big[K_1\cos(\omega_1\theta+\varphi_1)+K_2\cos(\omega_2\theta+\varphi_2)\big]+K_3\exp(-2\theta), \tag{5a}$$

$$\beta(t) = \exp(-5\theta/2)\big[(4+6\sqrt{6})K_1\cos(\omega_1\theta+\varphi_1) \tag{5b}$$

$$+(4-6\sqrt{6})K_2\cos(\omega_2\theta+\varphi_2)\big]+30K_3\exp(-4\theta), \tag{5c}$$

$$\gamma(t) = -4\exp(-9\theta/2)\big[(26+9\sqrt{6})K_1\cos(\omega_1\theta+\varphi_1) \tag{5d}$$

$$+(26-9\sqrt{6})K_2\cos(\omega_2\theta+\varphi_2)\big]+200K_3\exp(-6\theta)\,, \tag{5e}$$

where $\theta = \ln(t-t_0)$. The two frequencies read

$$\omega_1 = \frac{1}{2}\sqrt{95-24\sqrt{6}}, \qquad \omega_2 = \frac{1}{2}\sqrt{95+24\sqrt{6}}\,, \tag{6}$$

where K_1, K_2, K_3, φ_1, and φ_2 are constants.

Chaotic phase of BKL scenario

- The manifold $\mathcal{M}$ defined by $\{K_1, K_2, K_3, \varphi_1, \varphi_2\}$ is a submanifold of $\mathbb{R}^5$. Thus, (5) presents generic solution as the measure of $\mathcal{M}$ is nonzero.
- The relative perturbations $\alpha/\tilde{a}$, $\beta/\tilde{b}$, and $\gamma/\tilde{c}$ grow proportionally as $\exp(\frac{1}{2}\theta)$. The multiplier $1/2$ plays the role of Lyapunov exponent. Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) becomes chaotic.
- We expect that the solution to (1)–(2) for any initial conditions is chaotic as well.

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Chaotic phase of BKL scenario (cont)

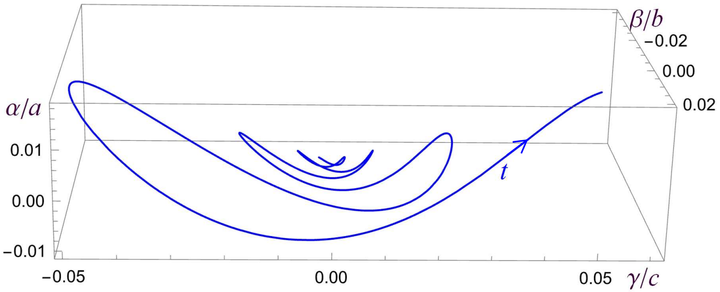

Figure: Linear instability of the special solution (3) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\varphi_1 = \varphi_2 = 0$ for the perturbations of the scale factors. The graph presents the parametric curve defined by the time dependence of $\alpha/\tilde{a}$, $\beta/\tilde{b}$, and $\gamma/\tilde{c}$.

Quantization of BKL scenario

In what follows, we quantize BKL scenario by making use of affine coherent states, ACS, quantization method (see, App A).
We have already quantized Hamilton's dynamics of that scenario ignoring its chaotic phase[9]:

- quantum singularity turns into quantum bounce
- quantum evolution is unitary across quantum bounce

For the rest of my talk I will present quantization of the chaotic phase of BKL scenario just presented earlier:

- we do not quantize Hamilton's dynamics, but the solution to BKL scenario
- we quantize both temporal and spatial variables to support general covariance of GR with respect to transformations of these variables

[9] A. Góźdź, W.P., and G. Plewa, Eur. Phys. J. C **79**, 45 (2019); A. Góźdź and W.P., Eur. Phys. J. C **80**, 142 (2020)

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Quantization of BKL scenario (cont)

Outline of calculations[10]

- We calculate expectation values and variances (see, App B) of quantum observables corresponding to perturbed $\{a, b, c\}$ and unperturbed $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ solutions to be compared.
- Expectation value of time operator is required to coincide with classical time.
- As quantum states, we take wave packets defined in the carrier (Hilbert) space of affine group representation.

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Stochastic aspects of quantum evolution

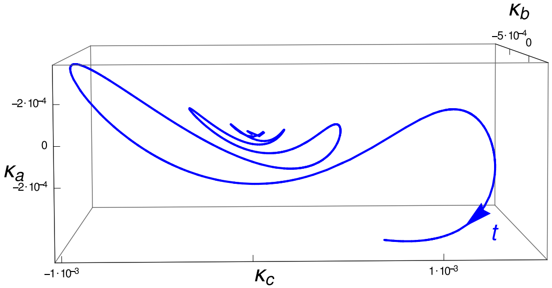

Figure: Typical parametric curve of relative quantum perturbations

$$\kappa_k := \frac{var(\hat{\xi}_k; \Psi_{pert}) - var(\hat{\xi}_k; \Psi_{unpert})}{var(\hat{\xi}_k; \Psi_{unpert})}, \qquad k = a, b, c \tag{7}$$

where $\hat{\xi}_a := \hat{a},\ \hat{\xi}_b := \hat{b},\ \hat{\xi}_c := \hat{c}.$

Conclusions

- The relative quantum perturbations grow as the system evolves towards the singularity.
- The quantum randomness amplifies the deterministic classical chaos.
- Hypothesis: In the region corresponding to the neighbourhood of the classical singularity the dynamics, both classical and quantum, enters a stochastic phase.
- Variances describe quantum smearing of observables; as calculated variances are always non-zero, the probability of obtaining divergencies corresponding to gravitational singularity is equal to zero[11].

[11] A quantum system is in eigenstate of given operator iff its variance equals zero.

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Thank you!

Essence of ACS quantization

The affine configuration space Π is a half-plane:

$$\Pi := \{(q,p) \in \mathbb{R} \times \mathbb{R}_+\}, \quad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}.$$

Π can be identified with the affine group $\mathrm{Aff}(\mathbb{R})$.
This group has UIR realized in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$, defined by

$$U(q,p)\psi(x) = e^{iqx}\psi(px).$$

This enables defining the continuous family of affine coherent states $|q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q,p\rangle = U(q,p)|\phi\rangle \ ,$$

where $|\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, is the so-called fiducial vector, which is a free parameter of ACS quantization scheme.

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This enables defining the continuous family of affine coherent states $|q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q,p\rangle = U(q,p)|\phi\rangle \ ,$$

where $|\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, is the so-called fiducial vector, which is a free parameter of ACS quantization scheme.

Essence of ACS quantization (cont

The irreducibility of the representation leads (due to Schur' lemma) to the resolution of the unity in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\int_\Pi d\mu(q,p)|q,p\rangle\langle q,p| = A_\phi\, \mathbb{I}\,, \tag{8}$$

where $d\mu(q,p) := dq\, dp/p^2$ is the left invariant measure on Π, and where $A_\phi := \int_0^\infty |\phi(x)|^2 \frac{dx}{x^2} < \infty$ is a constant.

Using (8), enables quantization of any observable $f : \Pi \to \mathbb{R}$

$$f \longrightarrow \hat{f} = \frac{1}{A_\phi} \int_\Pi d\mu(q,p)|q,p\rangle f(q,p)\langle q,p|\,. \tag{9}$$

The operator $\hat{f}$ is symmetric (Hermitian) by construction.
No ordering ambiguity occurs (disaster of canonical quantization).
For more details on ACS quantization see our recent paper[12].

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Variance of quantum observable

Variance is a stochastic deviation from expectation value of quantum observable; it determines the value of smearing of quantum observable[13].

The variance is the average of the squared differences from the mean. In the quantum state labelled by ψ, the variance is defined to be

$$var(\hat{A};\psi) := \langle(\hat{A} - \langle\hat{A};\psi\rangle)^2;\psi\rangle = \langle\hat{A}^2;\psi\rangle - \langle\hat{A};\psi\rangle^2 \,, \tag{10}$$

where $\langle\hat{B};\psi\rangle := \langle\psi|\hat{B}|\psi\rangle$.

If $\hat{A}$ is a self-adjoint operator, we have the important statement:

$$\Big(var(\hat{A};\psi) = 0\Big) \Longleftrightarrow \Big(\hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R}\Big) \,, \tag{11}$$

i.e., the variance of the operator $\hat{A}$ equals zero, if and only if, the quantum system is in an eigenstate of the operator $\hat{A}$.

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