# Noncommutative differential geometry and quantum effects of gravity

Haoyuan Gao/Xiao Zhang
Guangxi Center for Mathematical Research
Guangxi University
and
Institute of Mathematics, AMSS
Chinese Academy of Sciences

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# Deformation Quantization

The usual pointwise product of functions is replaced by the noncommutative star product

- Weyl (1931), Wigner (1932), Groenewold (1946), Moyal (1949)
  - Root ideas
- 1964, Gerstenhaber
  - Deformation of rings and algebras
- 1977, Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer
  - Definitive approach based on the deformation of algebra
- 1989, Fedosov
  - Deformation quantization of symplectic manifolds and index theory

- 1997, Kontsevich
  - Deformation quantization of Poisson manifolds, existence and classification of star products on Poisson manifolds
- 2005-2006, Aschieri, Blohmann, Meyer, Schupp, Wess
  - Noncommutative geometry and noncommutative gravity theory
- 2006, Chaichian, Tureanu, R.B. Zhang, X. Zhang
  - Deformation quantization of metrics and curvatures over a coordinate chart in manifolds, inspired by certain physical research
  - The theory is not covariant under coordinate transformations. And it fits the general feature of quantum world

# Basic Idea

- Let $(M, g)$ be an $n$-dimensional (pseudo-)Riemannian manifold. Let $U \subset M$ be a coordinate chart equipped with the local coordinate system $\{x^1, \cdots, x^n\}$
- Over $U$, the metric $g$ is written as

$$g = g_{ij} dx^i \otimes dx^j, \quad (g_{ij}) \text{ is a symmetric } n \times n \text{ matrix}$$

- Deform $g$ to $\mathbf{g}$ over $U$ by replacing usual function product to the Moyal product with constant skew-symmetric matrix $(\theta^{ij})$
- $\mathbf{g}$ is called a *quantum fluctuation* of $g$ over $U$

# Moyal Product

- Let $(\theta^{ij})$ be any constant skew-symmetric matrix. Let $\hbar$ be the Planck constant. The Moyal product of two smooth functions $u$ and $v$ is defined as

$$u * v = \lim_{x' \to x} \exp\left(\hbar\theta^{ij}\partial_i \otimes \partial_j'\right)u(x)v(x')$$
$$= uv + \hbar\theta^{ij}\partial_i u \partial_j v + \frac{\hbar^2}{2}\theta^{ik}\theta^{jl}\partial_{ij}u\partial_{kl}v + \cdots$$

which satisfies

(i) $(u * v) * w = u * (v * w)$
(ii) $[x^i, x^j] = 2\hbar\theta^{ij}$
(iii) Leibniz rule:

$$\partial_i(u * v) = (\partial_i u) * v + u * (\partial_i v)$$

# Noncommutative Metric

- From now on, matrix $(\theta^{ij})$ is assumed to be constant
- Let $\mathcal{A} = \{ \sum_{i\geq 0} f_i \hbar^i \}$ be the Moyal algebra on $U$ equipped with the Moyal product, where $f_i$ are smooth functions on $U$
- Denote $E_i = \tilde{E}_i = \partial_i$. The noncommutative left (right) tangent bundle $\mathcal{T}_U$ ($\tilde{\mathcal{T}}_U$) on $U$ is defined as

$$\mathcal{T}_U = \{ a^i * E_i \big| a^i \in \mathcal{A} \}, \qquad \tilde{\mathcal{T}}_U = \{ \tilde{E}_i * a^i \big| a^i \in \mathcal{A} \}$$

- A noncommutative metric $\mathbf{g}$ on $U$ is an $\mathcal{A}$-bilinear map

$$\mathcal{T}_U \otimes_{\mathbb{R}[[\hbar]]} \tilde{\mathcal{T}}_U \longrightarrow \mathcal{A}$$

such that the matrix $(\mathbf{g}_{ij})$ is invertible, where

$$\mathbf{g}_{ij} = \mathbf{g}(E_i, \tilde{E}_j)$$

- $(\mathbf{g}_{ij})$ is an $n \times n$ matrix which is not symmetric in general
- For any $V = v^i * E_i \in \mathfrak{T}_U$, $W = \tilde{E}_j * w^j \in \tilde{\mathfrak{T}}_U$

$$\mathbf{g}(V, W) = v^i * \mathbf{g}_{ij} * w^j$$

- There exists unique matrix $(\mathbf{g}^{ij})$ such that

$$\mathbf{g}_{ik} * \mathbf{g}^{kj} = \mathbf{g}^{ik} * \mathbf{g}_{kj} = \delta^i_j$$

- Define $E^i = \mathbf{g}^{ij} * E_j$, $\tilde{E}^j = \tilde{E}_i * \mathbf{g}^{ij}$. Then

$$\mathbf{g}(E^i, \tilde{E}_j) = \mathbf{g}^{ik} * \mathbf{g}(E_k, \tilde{E}_j) = \delta^i_j$$
$$\mathbf{g}(E_i, \tilde{E}^j) = \mathbf{g}(E_i, \tilde{E}_k) * \mathbf{g}^{kj} = \delta^j_i$$
$$\mathbf{g}(E^i, \tilde{E}^j) = \mathbf{g}^{ik} * \mathbf{g}(E_k, \tilde{E}_l) * \mathbf{g}^{lj} = \mathbf{g}^{ij}$$

- The noncommutative left (right) cotangent bundle $\mathcal{T}_U^*$ ($\tilde{\mathcal{T}}_U^*$) on $U$ is defined as

$$\mathcal{T}_U^* = \left\{ a_i * E^i \middle| a_i \in \mathcal{A} \right\}, \qquad \tilde{\mathcal{T}}_U^* = \left\{ \tilde{E}^i * a_i \middle| a_i \in \mathcal{A} \right\}$$

- $\mathbf{g}$ is an element of $\tilde{\mathcal{T}}_U^* \otimes_{\mathcal{A}} \mathcal{T}_U^*$, which can be written as

$$\tilde{E}^i \otimes \mathbf{g}_{ij} * E^j = \tilde{E}^i * \mathbf{g}_{ij} \otimes E^j$$

- $\mathbf{g}^{-1}$ is an element of $\tilde{\mathcal{T}}_U \otimes_{\mathcal{A}} \mathcal{T}_U$, which can be written as

$$\tilde{E}_i \otimes \mathbf{g}^{ij} * E_j = \tilde{E}_i * \mathbf{g}^{ij} \otimes E_j$$

# Noncommutative Left and Right Covariant Derivatives

- A noncommutative left covariant derivative $\nabla_i$ is a mapping

$$\nabla_i : \mathfrak{T}_U \longrightarrow \mathfrak{T}_U, \qquad \nabla_i E_j = \Gamma_{ij}^k * E_k$$

  which satisfies
  - Linearity: $\nabla_i \big(E_k + E_l\big) = \nabla_i E_k + \nabla_i E_l$
  - Leibniz rule: $\nabla_i \big(f * E_k\big) = \big(\partial_i f\big) * E_k + f * \nabla_i E_k, \quad f \in \mathcal{A}$
- A noncommutative right covariant derivative $\tilde{\nabla}_i$ is a mapping

$$\tilde{\nabla}_i : \tilde{\mathfrak{T}}_U \longrightarrow \tilde{\mathfrak{T}}_U, \qquad \tilde{\nabla}_i \tilde{E}_j = \tilde{E}_k * \tilde{\Gamma}_{ij}^k$$

  which satisfies
  - Linearity: $\tilde{\nabla}_i \big(\tilde{E}_k + \tilde{E}_l\big) = \tilde{\nabla}_i \tilde{E}_k + \tilde{\nabla}_i \tilde{E}_l$
  - Leibniz rule: $\tilde{\nabla}_i \big(\tilde{E}_k * f\big) = \tilde{E}_k * \big(\partial_i f\big) + \tilde{\nabla}_i \tilde{E}_k * f, \quad f \in \mathcal{A}$

# Induced Covariant Derivatives on Cotangent Bundles

- The induced noncommutative left covariant derivative is

$$\nabla_i : \mathcal{T}^*_U \longrightarrow \mathcal{T}^*_U, \qquad \nabla_i E^j = \Gamma^{*j}_{ik} * E^k$$

- The induced noncommutative right covariant derivative is

$$\tilde{\nabla}_i : \tilde{\mathcal{T}}^*_U \longrightarrow \tilde{\mathcal{T}}^*_U, \qquad \tilde{\nabla}_i \tilde{E}^j = \tilde{E}^k * \tilde{\Gamma}^{*j}_{ik}$$

- They are required to satisfy the compatible conditions

$$\partial_k \mathbf{g}(E_i, \tilde{E}^j) = \mathbf{g}(\nabla_k E_i, \tilde{E}^j) + \mathbf{g}(E_i, \tilde{\nabla}_k \tilde{E}^j) \Longrightarrow \tilde{\Gamma}^{*j}_{ik} = -\Gamma^j_{ik}$$
$$\partial_k \mathbf{g}(E^i, \tilde{E}_j) = \mathbf{g}(\nabla_k E^i, \tilde{E}_j) + \mathbf{g}(E^i, \tilde{\nabla}_k \tilde{E}_j) \Longrightarrow \Gamma^{*j}_{ik} = -\tilde{\Gamma}^j_{ik}$$

- Thus

$$\nabla_i E^j = -\tilde{\Gamma}^j_{ik} * E^k, \qquad \tilde{\nabla}_i \tilde{E}^j = -\tilde{E}^k * \Gamma^j_{ik}$$

# Covariant Derivative along Vector Fields

- For vector fields $V = a^i * E_i$ or $W = \tilde{E}_i * a^i$

$$\nabla_V E_j := a^i * \left(\nabla_i E_j\right), \qquad \tilde{\nabla}_W \tilde{E}_j := \left(\tilde{\nabla}_i \tilde{E}_j\right) * a^i$$

- However, they are not consistent with the Leibniz rule, e.g.,

$$\begin{aligned} \nabla_V\left(f * E_j\right) =& \left(Vf\right) * E_j + f * \nabla_V E_j \\ =& a^i * (\partial_i f) * E_j + f * a^i * \Gamma^k_{ij} * E_k \\ \nabla_V\left(f * E_j\right) =& a^i \nabla_i\left(f * E_j\right) \\ =& a^i * (\partial_i f) * E_j + a^i * f * \Gamma^k_{ij} * E_k \\ \Longrightarrow f * a^i * \Gamma^k_{ij} =& a^i * f * \Gamma^k_{ij} \end{aligned}$$

- This last equality may not hold in general. But it does hold if $f$ is constant
- It indicates that the noncommutative covariant derivative is not well-defined in terms of orthonormal basis

# Canonical Connection

- Call $(\nabla_1, \cdots, \nabla_n)$ a left connection and $(\tilde{\nabla}_1, \cdots, \tilde{\nabla}_n)$ a right connection
- Denote $\Gamma_{ijl} = \Gamma_{ij}^k * \mathbf{g}_{kl}$, $\tilde{\Gamma}_{ijl} = \mathbf{g}_{lk} * \tilde{\Gamma}_{ij}^k$
- A connection is *canonical* with respect to $\mathbf{g}$ and $\Upsilon$ if it satisfies the following conditions
  (i) compatibility: $\partial_k \mathbf{g}_{ij} = \mathbf{g}(\nabla_k E_i, \tilde{E}_j) + \mathbf{g}(E_i, \tilde{\nabla}_k \tilde{E}_j) = \Gamma_{kij} + \tilde{\Gamma}_{kji}$
  (ii) torsion free: $\nabla_i E_j = \nabla_j E_i, \tilde{\nabla}_i \tilde{E}_j = \tilde{\nabla}_j \tilde{E}_i \Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k, \tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{ji}^k$
  (iii) chirality: $\Gamma_{ijl} - \tilde{\Gamma}_{ijl} = \Upsilon_{ijl}$
- These conditions can determine the connection uniquely

$$\Gamma_{ijl} = \frac{1}{2}(\partial_i \mathbf{g}_{jl} + \partial_j \mathbf{g}_{li} - \partial_l \mathbf{g}_{ij}) + \frac{1}{2}(\Upsilon_{ilj} + \Upsilon_{jil} - \Upsilon_{lji})$$

# Covariant Derivative of Noncommutative Metric

- Define

$$\nabla_k \mathbf{g} := \tilde{E}^i \otimes \nabla_k \mathbf{g}_{ij} * E^j = \tilde{E}^i * \nabla_k \mathbf{g}_{ij} \otimes E^j$$

- By the Leibniz rule

$$\begin{aligned}\nabla_k \mathbf{g} =& \tilde{\nabla}_k \tilde{E}^i \otimes \mathbf{g}_{ij} * E^j + \tilde{E}^i \otimes \partial_k \mathbf{g}_{ij} * E^j + \tilde{E}^i \otimes \mathbf{g}_{ij} * \nabla_k E^j \\ =& - \tilde{E}^l * \Gamma^i_{kl} \otimes \mathbf{g}_{ij} * E^j + \tilde{E}^i \otimes \partial_k \mathbf{g}_{ij} * E^j \\ & - \tilde{E}^i \otimes \mathbf{g}_{ij} * \tilde{\Gamma}^j_{kl} * E^l \\ =& \tilde{E}^i \otimes \left(\partial_k \mathbf{g}_{ij} - \Gamma_{kij} - \tilde{\Gamma}_{kji}\right) * E^j\end{aligned}$$

- For metric compatible connections
  - It gives, $\nabla_k \mathbf{g}_{ij} = 0$
  - Same argument shows $\nabla_k \mathbf{g}^{ij} = 0$

# Curvatures of Canonical Connection

- $[E_i, E_j] = [\tilde{E}_i, \tilde{E}_j] = 0$
- Left and right Curvature tensors
  - $\mathcal{R}_{E_i E_j} := [\nabla_i, \nabla_j], \; \tilde{\mathcal{R}}_{\tilde{E}_i \tilde{E}_j} := [\tilde{\nabla}_i, \tilde{\nabla}_j]$
- Riemannian curvatures
  - Left: $\mathbf{R}_{lkij} = \mathbf{g}\big(\mathcal{R}_{E_i E_j} E_k, \tilde{E}_l\big)$
  - Right: $\tilde{\mathbf{R}}_{lkij} = -\mathbf{g}\big(E_k, \tilde{\mathcal{R}}_{\tilde{E}_i \tilde{E}_j} \tilde{E}_l\big) = \mathbf{R}_{lkij}$
  - $\mathbf{R}_{lkij} = -\mathbf{R}_{lkji}, \; \mathbf{R}_{lkij} \neq -\mathbf{R}_{klij}$
- Ricci curvatures: Two different Ricci curvatures are yielded by contracting $l, i$ and $k, j$ in $\mathbf{R}_{lkij}$ respectively
  - $\mathbf{R}_j^p := \mathbf{g}\big(\mathcal{R}_{E_i E_j} E^p, \tilde{E}^i\big) = \mathbf{g}^{pk} * \mathbf{R}_{lkij} * \mathbf{g}^{li}$
  - $\mathbf{\Theta}_i^p := \mathbf{g}\big(\mathcal{R}_{E_i E_j} E^j, \tilde{E}^p\big) = \mathbf{g}^{jk} * \mathbf{R}_{lkij} * \mathbf{g}^{lp}$
- Scalar curvature: The traces of $\mathbf{R}_j^i$ and $\mathbf{\Theta}_j^i$ equal, and they are defined as the scalar curvature
  - $\mathbf{R} := \mathbf{R}_i^i = \mathbf{\Theta}_j^j$

# Bianchi Identities for Canonical Connection

- The first Bianchi identity (algebraic identity)
  - $\mathcal{R}_{E_i E_j} E_k + \mathcal{R}_{E_j E_k} E_i + \mathcal{R}_{E_k E_i} E_j = 0$
- The second Bianchi identity (differential identity)
  - Define

$$\left(\nabla_k \mathcal{R}\right)_{E_i E_j} E_p := \nabla_k \left(\mathcal{R}_{E_i E_j}\right) E_p - \mathcal{R}_{(\nabla_{E_k} E_i) E_j} E_p - \mathcal{R}_{E_i (\nabla_{E_k} E_j)} E_p - \mathcal{R}_{E_i E_j} \left(\nabla_{E_k} E_p\right)$$

  - $\left(\nabla_k \mathcal{R}\right)_{E_i E_j} E_p + \left(\nabla_i \mathcal{R}\right)_{E_j E_k} E_p + \left(\nabla_j \mathcal{R}\right)_{E_k E_i} E_p = 0$
- Bianchi identities hold also for the right curvature tensor

# Quantum Fluctuation via Isometric Embedding

- Let $X = (X^1, \cdots, X^m)$ be the isometric embedding of $(U, g_{ij})$ into $(\mathbb{R}^{p,m-p}, \eta)$, $E_i = \partial_i X$, $\tilde{E}_i = \left(\partial_i X\right)^t$. Then

$$g_{ij} = \eta_{\alpha\alpha}\partial_i X^\alpha \cdot \partial_j X^\alpha = E_i \cdot \eta \tilde{E}_j$$

- The quantum fluctuation $\mathbf{g}$ can be obtained as follows

$$\mathbf{g}_{ij} = \eta_{\alpha\alpha}\partial_i X^\alpha * \partial_j X^\alpha = E_i * \eta \tilde{E}_j$$
$$\Upsilon_{ijl} = \partial_i E_j * \eta \tilde{E}_l - E_l * \eta \partial_i \tilde{E}_j$$

- The curvatures of $\mathbf{g}$ are completely determined

# Noncommutative Einstein Field Equations

- 2008, D. Wang, R.B. Zhang, X. Zhang proposed the noncommutative Einstein field equations

$$\mathbf{R}^i_j - \frac{1}{2}\delta_{ij}\mathbf{R} = \mathbf{T}^i_j, \qquad \mathbf{\Theta}^i_j - \frac{1}{2}\delta_{ij}\mathbf{R} = \tilde{\mathbf{T}}^i_j$$

  where $\mathbf{T}^i_j$, $\tilde{\mathbf{T}}^i_j$ are noncommutative energy-momentum tensors
- The vacuum field equations are

$$\mathbf{R}^i_j = \mathbf{\Theta}^i_j = 0$$

- The second Bianchi identity does not imply the conservation of the energy-momentum in noncommutative case

# Field Equations for Isometric Embedding

- 2022, H. Gao, X. Zhang
- If the quantum fluctuation $\mathbf{g}$ is given by an isometric embedding, then the two Ricci curvatures $\mathbf{R}^i_j$ and $\mathbf{\Theta}^i_j$ can determine each other completely
- The Einstein field equations can be reduced to

$$\mathbf{R}^i_j - \frac{1}{2}\delta_{ij}\mathbf{R} = \mathbf{T}^i_j$$

# Spherically Symmetric Embedding

- Let $U = (0, \infty) \times (0, 2\pi) \times (0, \pi) \times \cdots \times (0, \pi) \in \mathbb{R}^{n+1}$ with standard spherical coordinates $(r, \theta_1, \cdots, \theta_n)$
- An isometric embedding $X = (X^1, \cdots, X^l, X^{l+1}, \cdots, X^{l+n+1})$ of $U$ into $\mathbb{R}^{l+n+1}$ is called spherically symmetric if it has the form

$$
\begin{aligned}
X^1 &= f^1(r) \\
&\dots \\
X^l &= f^l(r) \\
X^{l+1} &= f^{l+1}(r)\mathrm{sin}\theta_\mathrm{n}\mathrm{sin}\theta_{\mathrm{n}-1}\cdots\mathrm{sin}\theta_2\mathrm{sin}\theta_1, \\
X^{l+2} &= f^{l+2}(r)\mathrm{sin}\theta_\mathrm{n}\mathrm{sin}\theta_{\mathrm{n}-1}\cdots\mathrm{sin}\theta_2\mathrm{cos}\theta_1, \\
&\dots \\
X^{l+n-1} &= f^{l+n-1}(r)\mathrm{sin}\theta_\mathrm{n}\mathrm{sin}\theta_{\mathrm{n}-1}\mathrm{cos}\theta_{\mathrm{n}-2}, \\
X^{l+n} &= f^{l+n}(r)\mathrm{sin}\theta_\mathrm{n}\mathrm{cos}\theta_{\mathrm{n}-1}, \\
X^{l+n+1} &= f^{l+n+1}(r)\mathrm{cos}\theta_\mathrm{n}
\end{aligned}
$$

# Renormalization of Spherically Symmetric Embedding

- 2022, H. Gao, X. Zhang
- Let $X = (X^1, \cdots, X^l, X^{l+1}, \cdots, X^{l+n+1})$ be a spherically symmetric embedding with $f^{l+1} = f^{l+2}$.
- For suitable matrix $(\theta^{ij})$ defining the Moyal product, the power series expansions of the quantum fluctuation $\mathbf{g}$ and curvatures $\mathbf{R}_{ijkl}, \mathbf{R}_j^i$ and $\mathbf{R}$ converge to smooth functions of variables $r, \theta_1, \cdots, \theta_n$ and $\hbar$
- Spherically symmetric embeddings are renormalizable

# Schwarzschild Spacetime

- $g_{Sch} = -\big(1 - \frac{2m}{r}\big)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2\big(d\theta^2 + \sin^2\theta d\psi^2\big)$
- Kasner's embedding (1921)

$$
\begin{aligned}
X^1 &= \big(1 - \frac{2m}{r}\big)^{\frac{1}{2}} \sin t \\
X^2 &= \big(1 - \frac{2m}{r}\big)^{\frac{1}{2}} \cos t \\
X^3 &= f(r), \quad (f')^2 + 1 = \big(1 - \frac{2m}{r}\big)^{-1}\big(1 + \frac{m^2}{r^4}\big) \\
X^4 &= r \sin\theta \cos\phi \\
X^5 &= r \sin\theta \sin\phi \\
X^6 &= r \cos\theta
\end{aligned}
$$

- $g_{Sch} = -(dX^1)^2 - (dX^2)^2 + (dX^3)^2 + (dX^4)^2 + (dX^5)^2 + (dX^6)^2$

- Fronsdal's embedding (1959)

$$X^1 = (1 - \frac{2m}{r})^{\frac{1}{2}} \sinh t$$
$$X^2 = (1 - \frac{2m}{r})^{\frac{1}{2}} \cosh t$$
$$X^3 = f(r), \quad (f')^2 + 1 = (1 - \frac{2m}{r})^{-1}(1 - \frac{m^2}{r^4})$$
$$X^4 = r \sin\theta \cos\phi$$
$$X^5 = r \sin\theta \sin\phi$$
$$X^6 = r \cos\theta$$

- $g_{Sch} = -(dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 + (dX^5)^2 + (dX^6)^2$
- The restriction on $t$-slices of these two embeddings are spherically symmetric and the results of general spherically symmetric embeddings make sense

# Quantum Fluctuation of Schwarzschild Spacetime

- 2008, D. Wang, R.B. Zhang, Z
- Choose $(\theta^{ij})$ such that $\theta^{23} = -\theta^{32} = 1$, others=0
- Two embeddings give rise to the same quantum fluctuation

$$
\begin{aligned}
\mathbf{g}_{00} &= -\left(1 - \frac{2m}{r}\right) \\
\mathbf{g}_{01} &= \mathbf{g}_{10} = \mathbf{g}_{02} = \mathbf{g}_{20} = \mathbf{g}_{03} = \mathbf{g}_{30} = 0 \\
\mathbf{g}_{11} &= \left(1 - \frac{2m}{r}\right)^{-1}\left[1 + \left(1 - \frac{2m}{r}\right)(\sin^2\theta - \cos^2\theta)\sinh^2\hbar\right] \\
\mathbf{g}_{12} &= \mathbf{g}_{21} = 2r\sin\theta\cos\theta\sinh^2\hbar \\
\mathbf{g}_{13} &= -\mathbf{g}_{31} = -2r\sin\theta\cos\theta\sinh\hbar\cosh\hbar \\
\mathbf{g}_{22} &= r^2\left[1 - \left(\sin^2\theta - \cos^2\theta\right)\sinh^2\hbar\right] \\
\mathbf{g}_{23} &= -\mathbf{g}_{32} = r^2\left(\sin^2\theta - \cos^2\theta\right)\sinh\hbar\cosh\hbar \\
\mathbf{g}_{33} &= r^2\left[\sin^2\theta + \left(\sin^2\theta - \cos^2\theta\right)\sinh^2\hbar\right]
\end{aligned}
$$

# Noncommutative Scalar Curvature

- The noncommutative scalar curvature $\mathbf{R}$ of $\mathbf{g}$ is

$$\mathbf{R} = \frac{A(r,\theta,\hbar)}{B(r,\theta,\hbar)}$$

$$A(r,\theta,0) = 0, \quad A(0,\theta,\hbar) \neq 0, \quad A(2m,\theta,\hbar) \neq 0,$$

$$B(r,\theta,\hbar) = 8r^2\Big[3r + 10m + (r-2m)(4\cosh 2\hbar + \cosh 4\hbar) - 16m\cos 2\theta \sinh^2\hbar\Big]^3$$

- It has the expansion

$$\mathbf{R} = \frac{2m\left[5r - 3m + 2(r-7m)\cos^2\theta\right]}{r^4}\hbar^2 + \frac{O(r)}{r^5}\hbar^4 + \cdots$$

# Un-Evaporated Quantum Black Hole

- $\mathbf{g}_{00}|_{r=2m} = 0$, $\mathbf{g}_{11}|_{r=2m} = \infty$, $\mathbf{R}|_{r=0} = \infty$, $\mathbf{R}|_{r=2m}$ is regular
  - Deformation quantization of the Schwarzschild black hole is a quantum black hole
- Both $\mathbf{g}$ and $\mathbf{R}$ do not depend on time $t$
  - This quantum black hole can not be evaporated
- $\mathbf{g}|_{\hbar=0} = g_{Sch}$
  - It implies that, in frame of deformation quantization, if the big band created quantum black holes, they could develop to the present classical black holes in the universe

# Plane-Fronted Gravitational Wave

- Brinkman (1925), Einstein-Rosen (1937), Ehlers-Kundt (1962)

$$g_{pp} = dx^2 + dy^2 + 2dudv + 2H(x, y, u)du^2$$

$$R^i_j = 0 \Longleftrightarrow H_{xx} + H_{yy} = 0$$

- Additivity: $(H_i)_{xx} + (H_i)_{yy} = 0 \Longrightarrow (\sum H_i)_{xx} + (\sum H_i)_{yy} = 0$
- Rosen (1965), Collinson (1968): isometric embedding

$$X = \left(x, y, \frac{Hu + u + v}{\sqrt{2}}, \frac{Hu - \frac{u^2}{2}}{\sqrt{2}}, \frac{Hu - u + v}{\sqrt{2}}, \frac{Hu + \frac{u^2}{2}}{\sqrt{2}}\right)$$

- $g_{pp} = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 - (dX^5)^2 - (dX^6)^2$

# Quantum Fluctuation of Plane-F. Gravitational Wave

- 2008, D. Wang, R.B. Zhang, X. Zhang
- Let $(\theta^{ij})$ be arbitrary constant skew symmetric $4 \times 4$ matrix
- The quantum fluctuation is

$$
\begin{aligned}
\mathbf{g}_{xx} =& \mathbf{g}_{yy} = \mathbf{g}_{uv} = \mathbf{g}_{vu} = 1 \\
\mathbf{g}_{xy} =& \mathbf{g}_{yx} = \mathbf{g}_{xv} = \mathbf{g}_{vx} = \mathbf{g}_{yv} = \mathbf{g}_{vy} = 0 \\
\mathbf{g}_{xu} =& -\mathbf{g}_{ux} = -\hbar(\theta_{yu} H_{xy} + \theta_{xu} H_{xx}) \\
\mathbf{g}_{yu} =& -\mathbf{g}_{uy} = -\hbar(\theta_{yu} H_{yy} + \theta_{xu} H_{xy}) \\
\mathbf{g}_{uu} =& 2H
\end{aligned}
$$

- $\mathbf{R}_3^4 = \mathbf{\Theta}_3^4 = -H_{xx} - H_{yy}, \quad \text{others} = 0$
- Exact solution: Vacuum $\Longleftrightarrow H_{xx} + H_{yy} = 0$
- Additivity holds also

# Thanks!