Hamiltonian energy of weak gravitational fields with a cosmological constant Λ

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Linearized gravity

- New description of canonical charges of gravitational waves emitted by an isolated system.
- Asymptotic conditions on linearised fields have been modeled on the asymptotic behavior of the full solutions of the Einstein equations $(\Lambda > 0)$ with smooth initial data.
- All boundary terms are taken into consideration.

de Sitter background in Bondi coordinates

- Metric in Bondi coordinates

$$g_{\alpha\beta}dx^\alpha dx^\beta = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big)\,, \quad (1)$$

where $(x^A) \equiv (x^2, x^3) \equiv (\theta, \varphi)$, together with the condition

$$\det\gamma_{AB} = \sin^2\theta\,. \quad (2)$$

- The de Sitter metric in this form

$$g_{ab}dx^a dx^b = \underbrace{-(1-\Lambda r^2/3)}_{=:\pm N^2} du^2 - 2du\,dr + r^2\underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{=:\mathring{\gamma}}\,, \quad (3)$$

- The gauge conditions for linearised fields

$$h_{rr} = 0 = h_{rA}\,, \quad \mathring{\gamma}^{AB}h_{AB} = 0\,. \quad (4)$$

Lagrangian and canonical energy

- From $L = \frac{\sqrt{|\det g|}}{16\pi}(R - \frac{\Lambda}{2})$, we obtain the Lagrangian for weak fields

$$\mathcal{L}[h] = \frac{1}{32\pi}\sqrt{|\det g|}\big(P^{\alpha\beta\gamma\delta\epsilon\sigma}\nabla_\alpha h_{\beta\gamma}\nabla_\delta h_{\epsilon\sigma} + Q(h)\big)\,, \tag{5}$$

where Q is a quadratic polynomial in h, and

$$\begin{aligned} P^{\alpha\beta\gamma\delta\epsilon\sigma} &= \frac{1}{2}\big(g^{\alpha\epsilon}g^{\delta\beta}g^{\gamma\sigma} + g^{\alpha\epsilon}g^{\sigma\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\epsilon}g^{\sigma\gamma} - g^{\alpha\beta}g^{\gamma\delta}g^{\epsilon\sigma} \\ &\quad - g^{\beta\gamma}g^{\alpha\epsilon}g^{\sigma\delta} + g^{\beta\gamma}g^{\alpha\delta}g^{\epsilon\sigma}\big)\,. \end{aligned} \tag{6}$$

- For a given Lagrangian, the canonical energy is defined as

$$\mathcal{H}[\mathcal{S}, X, \phi] := \int_{\mathcal{S}} \underbrace{\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu h_{\nu\lambda})}\pounds_X h_{\nu\lambda} - X^\mu\mathcal{L}\Big)}_{=:\mathcal{H}^\mu} dS_\mu\,. \tag{7}$$

Symplectic charge[1]

Consider a solution ϕ of the linearised field equations and assume that the vector field X is independent of the fields considered. The Hamiltonian current

$$\mathcal{H}^\mu[X] \quad := \quad \pi^\mu{}_A \pounds_X \phi^A - X^\mu \mathcal{L} \tag{8}$$

of the linearised theory can be rewritten as

$$\mathcal{H}^\mu[X] \quad = \quad \frac{1}{2}\omega^\mu(\phi, \pounds_X\phi) + \partial_\sigma\Big(X^{[\sigma}\pi_A{}^{\mu]}\phi^A\Big)\,. \tag{9}$$

Here $\mathcal{L}$ is the Lagrangian density for the linearised equations, with $\pi^A{}_\mu = \partial\mathcal{L}/\partial(\partial_\mu\phi^A)$, and $\omega^\mu(\phi, \pounds_X\phi) := \pounds_X\phi^A\pi_A{}^\mu - \phi^A\pounds_X\pi_A{}^\mu$ is the presymplectic current.

[1] Chruściel, P. T., Hoque, S. J., Maliborski, M., Smołka, T. (2021). EPJC 81(8), 1-48.

Symplectic charges

- Consider symplectic charges associated with ten-dimensional space of Killing vector fields.
- In each charge, we observe the boundary term leads to divergences

$$\mathcal{H}^{\mu}[X] \quad = \quad \frac{1}{2}\omega^{\mu}(\phi, \pounds_X \phi) + \partial_\sigma \left(X^{[\sigma} \pi_A{}^{\mu]} \phi^A \right) . \tag{10}$$

- Renormalisation procedure will be discussed by the example of energy and its flux.

Example: Canonical energy in Bondi gauge

The canonical energy $E_c[h, \mathcal{C}_{u,R}]$ is given by

$$\begin{aligned} E_c[h, \mathcal{C}_{u,R}] &:= \mathcal{H}[\mathcal{C}_{u,R}, \partial_u, h] \\ &= \frac{1}{64\pi} \int_{\mathcal{C}_{u,R}} g^{BE} g^{FC} \left(\partial_u h_{BC} \partial_r h_{EF} - h_{BC} \partial_r \partial_u h_{EF}\right) r^2 \sin\theta \, dr \, d\theta \, d\varphi \\ &\quad - \frac{1}{32\pi} \int_{S(R)} P^{r(\beta\gamma)\delta(\epsilon\sigma)} \nabla_\delta h_{\epsilon\sigma} \, h_{\beta\gamma} \, r^2 \sin\theta \, d\theta \, d\varphi \,, \end{aligned} \tag{11}$$

where:

$h_{\mu\nu}$	solution of the linearised vacuum Einstein equations,
$\mathcal{C}_u$	light cone $u = \mathrm{const}$ emanating from $r = 0$,
$\mathcal{C}_{u,R}$	light cone truncated at radius $r = R$,
$S(R)$	sphere of radius R.

Asymptotic behavior for large r

- There exists a dynamically consistent class of fields[2] $h_{AB} = r^2 \check{h}_{AB}$ which have an asymptotic expansion of the form

$$\check{h}_{AB} = \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \ldots \tag{12}$$

[2] H. Friedrich, *On the existence of n–geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure,* Commun. Math. Phys. **107** (1986), 587-609.

Boundary term in canonical energy

The remaining $h_{\mu\nu}$'s are determined by the linearised version of constraint equations in Bondi coordinates:

$$
\begin{aligned}
h_{ru} &\equiv 0\,, && (13)\\
\check{h}_{uA} &= \overset{(0)}{\check{h}}_{uA} + \overset{(2)}{\check{h}}_{uA}/r^2 + \ldots\,, && (14)\\
\check{h}_{AB} &= \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \ldots\,. && (15)
\end{aligned}
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One finds the following form of the boundary term in $E_c[h, \mathcal{C}_{u,R}]$:

$$-\frac{\Lambda R}{192\pi}\int_{S^2} \mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\overset{(1)}{\check{h}}_{BD}\sin(\theta)\,d\theta\,d\varphi$$
$$-\frac{1}{64\pi}\int_{S^2}\mathring{\gamma}^{AB}\left(\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD} - 6\overset{(0)}{\check{h}}_{uA}\overset{(3)}{\check{h}}_{uB}\right)\sin(\theta)\,d\theta\,d\varphi$$
$$+o(1)\,, \tag{16}$$

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1. the divergence of the boundary integral is compensated by that of the volume integral and, if not,
2. whether the boundary integral is needed at all in the definition of energy and, if so,
3. can one obtain consistent solutions by restricting oneself to a set of fields with $\overset{(1)}{\check{h}}_{AC} \equiv 0$.

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&+o(1)\,,
\end{aligned}
\tag{13}
$$

Linearised evolution equation for g_{AB}

Denoting by $TS[\cdot]$ the traceless symmetric part of a tensor, we have in vacuum

$$r\partial_r[r(\partial_u \check{h}_{AB})] - \frac{1}{2}\partial_r[N^2(\partial_r \check{h}_{AB})] + TS\big[\mathring{D}_A\big(\partial_r(r^2\check{h}_B)\big)\big] = 0\,, \tag{17}$$

where $N^2 = \pm(1 - \Lambda r^2/3)$. Integrating, we find

$$\begin{aligned} \partial_u \check{h}_{AB}(r,\cdot) &= \frac{1}{r}\int_0^r \frac{1}{s}\Big(\frac{1}{2}\partial_r[N^2(\partial_r \check{h}_{AB})] - TS\big[\mathring{D}_A\big(\partial_r(r^2\check{h}_B)\big)\big]\Big)(s,\cdot)ds \\ &= \frac{\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot)}{r} + \frac{\partial_u \overset{(3)}{\check{h}}_{AB}(\cdot)}{r^3} + o(r^{-3})\,, \end{aligned} \tag{18}$$

where

$$\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot) = \int_0^\infty \frac{1}{s}\Big(\frac{1}{2}\partial_r[N^2(\partial_r \check{h}_{AB})] - \mathring{\gamma}_{CA}\mathring{D}_B[\partial_r(r^2\check{h}^C)]\Big)(s,\cdot)ds\,, \tag{19}$$

Volume integral from $E_c[h, \mathcal{C}_{u,R}]$

Using the obtained field behavior

$$\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot) = \frac{\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot)}{r} + \frac{\partial_u \overset{(3)}{\check{h}}_{AB}(\cdot)}{r^3} + o(r^{-3}) \,, \tag{20}$$

$$\check{h}_{AB} = \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \ldots \,, \tag{21}$$

one finds a finite volume contribution to the canonical energy

$$\int_{\mathcal{C}_{u,R}} g^{BE} g^{FC} \left(\partial_u h_{BC} \partial_r h_{EF} - h_{BC} \partial_r \partial_u h_{EF}\right) r^2 \sin\theta \, dr \, d\theta \, d\varphi \tag{22}$$

Energy flux formula

The flux formula is also divergent

$$
\begin{aligned}
\frac{dE_c[h,\mathcal{C}_{u,R}]}{du} &= \\
&-\frac{\Lambda R}{96\pi}\int_{S^2}\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD}\,\sin(\theta)\,d\theta\,d\varphi \\
&-\frac{1}{32\pi}\int_{S^2}\mathring{\gamma}^{AB}\left(\mathring{\gamma}^{CD}\partial_u\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD}-6\overset{(3)}{\check{h}}_{uA}\partial_u\overset{(0)}{\check{h}}_{uB}\right)\sin(\theta)\,d\theta\,d\varphi \\
&+o(1)\,. \qquad (23)
\end{aligned}
$$

Renormalised energy and flux

We propose to introduce a *renormalised canonical energy*, obtained by removing the divergent terms in the canonical energy

$$\hat{E}_c[h, \mathcal{C}_u] := \frac{1}{64\pi} \int_{\mathcal{C}_u} g^{BE} g^{FC} \left(\partial_u h_{BC} \partial_r h_{EF} - h_{BC} \partial_r \partial_u h_{EF} \right) r^2 \, dr \sin(\theta) \, d\theta \, d\varphi - \frac{1}{64\pi} \int_{S^2} \mathring{\gamma}^{AB} \left(\mathring{\gamma}^{CD} \overset{(1)}{\check{h}}_{AC} \partial_u \overset{(1)}{\check{h}}_{BD} - 6 \overset{(0)}{\check{h}}_{uA} \overset{(3)}{\check{h}}_{uB} \right) \sin(\theta) \, d\theta \, d\varphi \, . \tag{24}$$

which has its own finite flux formula

$$\frac{d\hat{E}_c[h, \mathcal{C}_{u,R}]}{du} = - \frac{1}{32\pi} \int_{S^2} \mathring{\gamma}^{AB} \left(\mathring{\gamma}^{CD} \partial_u \overset{(1)}{\check{h}}_{AC} \partial_u \overset{(1)}{\check{h}}_{BD} - 6 \overset{(3)}{\check{h}}_{uA} \partial_u \overset{(0)}{\check{h}}_{uB} \right) \sin(\theta) \, d\theta \, d\varphi \, . \tag{25}$$

For $\Lambda = 0$, we recover the weak-field version of the usual Trautman-Bondi mass loss formula.

Maxwell theory

- Smooth Maxwell field have expansions of the form

$$F_{Ar} \quad = \quad \overset{(2)}{F}_{Ar} r^{-2} + \dots \,, \qquad (26)$$

where $\overset{(i)}{F}_{Ar} = \overset{(i)}{F}_{Ar} (u, x^A)$.

- $\nabla_\nu F^{\mu\nu} = 0 \,, \nabla_\mu * F^{\mu\nu} = 0 \,,$ restrict asymptotics of the other components of $F_{\mu\nu}$.
- All Hamiltonian charges $\int_{\mathcal{C}_u} \mathcal{H}^\mu[X, F] dS_\mu$ and their fluxes are well defined,

$$\mathcal{H}^\mu[X] \quad := \quad \pi^{\mu\beta} \mathbf{L}_X A_\beta - X^\mu \mathcal{L} \qquad (27)$$

where $\mathbf{L}_X A_\mu := X^\nu F_{\nu\mu}$ and $\mathcal{L} = -\frac{1}{16\pi} \sqrt{|g|} F^{\mu\nu} F_{\mu\nu} \,.$

Summary and references[3]

- The asymptotic conditions satisfied by the linearized metric have been modeled on the asymptotic behavior of the full solutions of the Einstein equations with positive cosmological constant.
- Near de Sitter spacetime, the canonical charges of weak gravitational fields in Bondi gauge are divergent in general.
- Proposed renormalised energy and flux in the limit $\Lambda = 0$ become classical Trautman-Bondi quantities.

Thank for your attention!

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[3] Chruściel, P. T., Hoque, J., Smołka, T. (2021). *Energy of weak gravitational waves in spacetimes with a positive cosmological constant.* PRD, 103(6), 064008. Chruściel, P. T., Hoque, J., Maliborski, M., Smołka, T. (2021). *On the canonical energy of weak gravitational fields with a cosmological constant* $\Lambda \in \mathbb{R}$. EPJC, 81(8), 1-48.