Affine Connections, Quantum Gravity and Modified Theories of Gravity

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- The talk is based on:
- Kaushik Ghosh, *Physics of the Dark Universe*, **26** (2019) 100403.
- An updated version is available at: K. Ghosh, arXiv:1910.12633v4 [gr-qc]
- Other references mentioned in the talk are:
- K. Sundermeyer, Constrained Dynamics (Springer-Verlag, 1995).
- D. Lovelock and H. Rund; *Tensors, Differential Forms, and Variational Principals* (Dover Publications, Inc., New York, 1989)

- Present data from different sources, such as the:
  Cosmic Microwave Background Radiation (CMBR) and supernovae surveys,
  seem to indicate that the energy composition of the universe consists of 20% dark matter, 76% dark energy and the rest ordinary baryonic matter.
- This together with inflation lead to search for fields that may not come from the standard model.
- The simplest candidate for dark energy is the cosmological constant Λ with constant energy density although it can not lead to a dynamical theory for dark energy, [Kobayashi and Nomizu, Vol I].
- There are two approaches to explain cosmic acceleration as alternatives to the cosmological constant model.

- The first is to supplement the source stress-tensor part of Einstein's equation by specific forms of stress-tensor with negative pressure.
- Among various models, cosmons or quintessence, k-essence and perfect fluid models are mostly studied.

- The second approach to explain dark energy is to modify the gravitational part of Einstein's equation.
- Examples are the so-called $f(R)$ gravity, scalar-tensor theories and braneworld models.

- In this presentation, we will try to find if quantum gravity can introduce additional fields besides metric.
- We will discuss this with reference to the construction of affine connections in quantum gravity.

- We briefly consider quantization of gravity by using the canonical quantization procedure.
- Canonical quantization is important to find the particle spectrum when we quantize a classical theory.
- In the canonical quantization of gravity, metric becomes operator on a Hilbert space.
- We now express covariant derivative operator in the following form:

$$\hat{\nabla}'_{\mu}\hat{A}_{\nu} = [\partial_{\mu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu}(\partial\hat{g}, \hat{g})]\hat{A}_{\alpha} \tag{1}$$

- Here, $\hat{\Gamma}^{\alpha}{}_{\mu\nu}$ are operator version of the Levi-Civita connections that in general contain conjugate operators and ordering of the operators are as shown above.

- We expect that:

$$[q^{\mu}\hat{\nabla}'_{\mu}q^{\nu}]\,|\Psi\rangle \neq 0 \tag{2}$$

will remain valid in a given state $|\Psi\rangle$ with an arbitrary well-behaved vector field q^{μ}.

- We will not have a complete set of states for which the expectation value of the operator in the *l.h.s* is zero with negligible fluctuations for all well-behaved vector fields. This will be valid only in the classical limit.
- Thus, the notion of parallel transport is not exact in a quantum theory of gravity. This is expected and indicates that we can use affine connections more general than the metric compatible connections even in free quantum gravity.

- Similar conclusion can be inferred if we consider the metric-metric commutator:

$$[\hat{g}_{\alpha\beta}(x^{\mu}), \hat{g}_{\mu\nu}(y^{\nu})] = iD_{\alpha\beta,\mu\nu}(x^{\mu}, y^{\nu}, \hat{g}) \quad (3)$$

- In the case of gravity, general commutators: $[\hat{g}_{\alpha\beta}(x^{\mu}), \hat{g}_{\alpha\beta}(y^{\nu})]$ will depend on (x^{μ}, y^{ν}) nontrivially.
- It is appropriate to use connections more general than metric compatible connections in the quantum theory so that the covariant derivatives of both sides agree.

- To introduce nonmetricity, we consider the metric-affine theory of gravity. The source-free action is given by:

$$S = \int \sqrt{-g} R \mathbf{e} \tag{4}$$

where, g is the determinant of metric, $\sqrt{-g}\mathbf{e}$ is the natural volume element associated with metric and R is the scalar curvature.

- In this formalism, both metric and affine connections are the independent variables. We have the following expression for covariant derivative operator:

$$\nabla_\mu A_\nu = \nabla'_\mu A_\nu - C^\alpha_{\ \mu\nu} A_\alpha \tag{5}$$

- ∇'_μ is torsion-free covariant derivative given by the Levi-Civita connections.

- With this choice of ∇'_μ, $C^\alpha_{\ \mu\nu}$ is a third rank tensor that can be asymmetric in the lower indices.
- ∇'_μ commutes with index raising/lowering operations.
- In the metric-affine theory, metric and respective $C^\alpha_{\ \mu\nu}$ are taken to be the independent variables.

- The Riemann curvature tensor is now defined by the following expressions:

$$\begin{aligned}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A^\alpha_{\ \beta} &= -R_{\mu\nu\kappa}{}^{\alpha}A^\kappa_{\ \beta} + R_{\mu\nu\beta}{}^{\kappa}A^\alpha_{\ \kappa} - T^\kappa_{\ \mu\nu}\nabla_\kappa A^\alpha_{\ \beta} \quad (6)\\ R_{\mu\nu\alpha}{}^{\kappa} &= R'{}_{\mu\nu\alpha}{}^{\kappa} + 2\nabla'{}_{[\nu}C^\kappa_{\ \mu]\alpha} + 2[C^\lambda_{\ [\mu|\alpha|}C^\kappa_{\ \nu|\lambda|]}]\\ T^\kappa_{\ \mu\nu} &= C^\kappa_{\ \mu\nu} - C^\kappa_{\ \nu\mu}\end{aligned}$$

- Here, $R'{}_{\mu\nu\alpha}{}^{\kappa}$ is the Riemann curvature tensor associated with the derivative ∇'_μ with the Levi-Civita connections, and is given by the familiar expression in terms of ordinary partial derivatives of $\Gamma^\alpha_{\ \mu\nu}$.

- $T^\kappa_{\ \mu\nu}$ is torsion tensor. The second equation is always valid when $\Gamma^\alpha_{\ \mu\nu}$ is symmetric in the lower indices. The curvature scalar is obtained by usual contractions.

- The Ricci tensor, scalar curvature and metric-affine action are given by the following expressions when we use affine connections given by Eqn.(5):

$$
\begin{aligned}
R_{\mu\alpha} &= R'_{\mu\alpha} + 2\nabla'_{[\kappa} C^{\kappa}_{\ \mu]\alpha} + 2[C^{\lambda}_{\ [\mu|\alpha|} C^{\kappa}_{\ \kappa|\lambda|]}] \\
R &= R' + 2g^{\mu\alpha} \left\{ \nabla'_{[\kappa} C^{\kappa}_{\ \mu]\alpha} + [C^{\lambda}_{\ [\mu|\alpha|} C^{\kappa}_{\ \kappa|\lambda|]}] \right\} \\
S &= \int \sqrt{-g} R\mathbf{e} + \kappa_M S_M(\psi, g_{\mu\nu})
\end{aligned}
\qquad (7)
$$

- where, $R'_{\mu\alpha}$ is the Ricci tensor evaluated using the Levi-Civita connections and R' is the corresponding scalar curvature.
- ψ is matter field and the matter field action $S_M(\psi, g_{\mu\nu})$ usually does not contain $C^{\alpha}_{\mu\nu}$.
- κ_M is a constant depending on the nature of source. In this talk, by matter fields we will mean both matter and gauge fields unless otherwise stated.

- We have the following equations as the solution of variational problem with respect to $C^{\alpha}_{\ \mu\nu}$ when these fields and their first-derivatives are held fixed at the boundaries:

$$C^{\kappa}_{\ \kappa\lambda} g^{\mu\alpha} + C^{\alpha\kappa}_{\ \ \kappa} \delta^{\mu}_{\lambda} - C^{\mu\alpha}_{\ \ \lambda} - C^{\alpha\ \mu}_{\ \lambda} = 0 \tag{8}$$

- There is no contribution from the source fields in the present case. This is a set of algebraic equations giving constraints on $C^{\alpha}_{\ \mu\nu}$ that can not determine $C^{\alpha}_{\ \mu\nu}$ uniquely in the metric-affine theory due projective invariance: $C^{\alpha}_{\ \mu\nu} \to C^{\alpha}_{\ \mu\nu} + \delta^{\alpha}_{\ \nu}\xi_{\mu}$, where ξ_{μ} is a regular covariant vector field (Hehl et al).
- This no longer remains valid for the Palatini formalism where we only consider symmetric $C^{\alpha}_{\ \mu\nu}$ and Eqn.(8) yields null solutions.

- The above formalism will give non-vanishing solutions for $C^{\alpha}_{\ \mu\nu}$ in presence of sources that couple with $C^{\alpha}_{\ \mu\nu}$. This is the case in the metric-affine gravity with fermions if we consider locally Lorentz invariant theory of quantum fields in curved spaces. Eq.(8) is replaced by the following expression:

$$
\begin{aligned}
C^{\kappa}_{\ \kappa\lambda} g^{\mu\alpha} + C^{\alpha\kappa}_{\ \ \kappa} \delta^{\mu}_{\lambda} - C^{\mu\alpha}_{\ \ \lambda} - C^{\alpha\ \mu}_{\ \lambda} &= \Delta_{\lambda}^{\ \mu\alpha} \\
\Delta_{\lambda}^{\ \mu\alpha} &= -\kappa_M \frac{\delta S_M(\psi, g_{\mu\nu}, C^{\lambda}_{\ \mu\nu})}{\delta C^{\lambda}_{\ \mu\nu}}
\end{aligned}
\tag{9}
$$

- where, $\Delta_{\lambda}^{\ \mu\alpha}$ is known as the hypermomentum [Hehl].

- However, matter field Lagrangians are usually polynomials in first order covariant derivatives. In addition, gauge fields do not couple with $C^{\alpha}_{\ \mu\nu}$ to express the corresponding field-strength tensors as gauge-covariant curls [Hehl]. Thus, the above equation remains an algebraic constraint. We also note that fermions introduce metric-compatible torsion and can not introduce non-metricity.
- It is required to extend the metric-affine and Palatini formalisms in quantum gravity to have finite and dynamical $C^{\alpha}_{\ \mu\nu}$ even in the **free theory**. Well-known candidates are metric-affine and Palatini $f(R)$ gravity.
- In the following, we will discuss a new approach that can introduce new dynamical fields.

- We can construct a theory by using the potential formalism. By potential formalism we mean a formalism where $C^{\alpha}_{\ \mu\nu}$ is derived from a tensor of lower rank. This is case with the Levi-Civita connections that are derived from metric.
- We can then have a set of differential equations including time derivatives in place of Eqns.(8,9) and we can introduce non-metricity even in the source-free theory by using nontrivial solutions of the homogeneous differential equations.
- This is similar to the four-wave solutions of the vacuum Einsteins equations.

- We start with a special case of the Palatini formalism where $\tilde{C}^{\alpha}_{\ \mu\nu}$ are given by the following expression [Lovelock and Hund]:

$$\begin{aligned} S^{\alpha}_{\ \mu\nu} &= \Gamma^{\alpha}_{\ \mu\nu} + \tilde{C}^{\alpha}_{\ \mu\nu} \\ &= \frac{1}{2}[\partial_{\mu}(b_{\kappa\nu}) + \partial_{\nu}(b_{\mu\kappa}) - \partial_{\kappa}(b_{\mu\nu})]\tilde{b}^{\alpha\kappa} \end{aligned} \tag{10}$$

- Where, $\Gamma^{\alpha}_{\ \mu\nu}$ are the Levi-Civita connections. $\tilde{C}^{\alpha}_{\ \mu\nu}$ is the symmetric in the lower indices. $b_{\mu\nu}$ is a non-singular symmetric covariant tensor and can be expressed as:

$$b_{\mu\nu} = g_{\mu\nu} + a_{\mu\nu} \tag{11}$$

- The inverse of $b_{\mu\nu}$, $\tilde{b}^{\mu\nu}$, is a contravariant tensor and can be expressed as $\tilde{b}^{\mu\nu} = g^{\mu\nu} + d^{\mu\nu}$.

- Note that $\tilde{b}^{\mu\nu}$ is different from $b^{\mu\nu}$ and $S^{\alpha}_{\ \mu\nu}$ satisfy the compatibility conditions: $b_{\mu\nu|\alpha} = 0$, where the bar denotes covariant derivative with connections $S^{\alpha}_{\ \mu\nu}$.
- It can be shown that the Ricci tensor is symmetric when affine connections are given by Eq.(10).
- Before we proceed further, we note that the action in Eqn.(7) give us the following equations:

- when we extremize the action *w.r.t* $g_{\mu\nu}$:

$$\begin{aligned} \mathcal{G}_{(\mu\alpha)} &= \mathcal{R}_{(\mu\alpha)} - \frac{1}{2}\mathcal{R}g_{\mu\alpha} = 8\pi P_{\mu\alpha} \\ \mathcal{R}_{\mu\alpha} &= R'_{\mu\alpha} + 2[C^{\lambda}{}_{[\mu|\alpha|}C^{\kappa}{}_{\kappa|\lambda|]}] \\ \mathcal{R} &= g^{\mu\alpha}\mathcal{R}_{\mu\alpha} \end{aligned} \tag{12}$$

  where, $\mathcal{R}_{\mu\nu}$ and $\mathcal{G}_{\mu\nu}$ are the modified Ricci tensor and modified Einstein tensor respectively.

- They differ from $R_{\mu\alpha}$ and $G_{\mu\alpha}$ respectively.

- $P_{\mu\alpha}$ is matter field stress tensor which is related to the variational derivative of matter field action *w.r.t* $g^{\mu\alpha}$ by: $8\pi P_{\mu\alpha} = -\frac{\kappa_M}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\alpha}}$. Here δ denotes functional derivatives and $P_{\mu\alpha}$ is symmetric.

- We have mentioned that we can introduce non-metricity by defining the affine connections to be compatible with a symmetric covariant field: $b_{\mu\nu} = g_{\mu\nu} + a_{\mu\nu};\ a_{\mu\nu} \neq k g_{\mu\nu}$, where k is a constant including zero.
- Such a set of affine connections give a symmetric Ricci tensor.
- We now break $a_{\mu\nu}$ into a trace and a traceless part:

$$a_{\mu\nu}(x) = \Phi(x) g_{\mu\nu} + \bar{a}_{\mu\nu}(x);\ \ \Phi(x) = \frac{a(x)}{4} \tag{13}$$

- Where, Φ is a scalar field, $a(x)$ is the trace of $a_{\mu\nu}$ and $\bar{a}_{\mu\nu}$ is trace-free.

- We can express corresponding $\tilde{C}^{\alpha}_{\mu\nu}$ in the following way:

$$\begin{aligned} \tilde{C}^{\alpha}_{\mu\nu} &= \delta^{\alpha}_{(\mu}\nabla'_{\nu)}[ln(1+\Phi)] - \frac{1}{2}g_{\mu\nu}\nabla'^{\alpha}[ln(1+\Phi)] \\ &+ D^{\alpha}{}_{\mu\nu} \\ &= \delta^{\alpha}_{(\mu}\nabla'_{\nu)}[ln(1+\Phi)] - \frac{1}{2}g_{\mu\nu}\nabla'^{\alpha}[ln(1+\Phi)] \\ &+ g_{\mu\nu}H^{\alpha} + E^{\alpha}{}_{\mu\nu} \\ &= \delta^{\alpha}_{(\mu}\nabla'_{\nu)}[ln(1+\Phi)] - \frac{1}{2}g_{\mu\nu}\nabla'^{\alpha}[ln(1+\Phi)] \\ &+ g_{\mu\nu}\nabla'^{\alpha}\Psi + g_{\mu\nu}B^{\alpha} + E^{\alpha}{}_{\mu\nu} \end{aligned} \tag{14}$$

- $E^{\alpha}{}_{\mu\nu}$ is traceless in the lower indices.
- The first two terms in the *r.h.s* gives the contribution of the trace part of $b_{\mu\nu}$ given by: $(g_{\mu\nu} + \Phi(x)g_{\mu\nu})$.

- B^α is a vector field with: $\nabla'_\alpha B^\alpha = 0$, and is suitable to describe a spin one boson. However, we also have: $\nabla'_{[\mu} B_{\nu]} = 0$, which follows from the symmetry of the Ricci tensor that remains valid with a symmetric $a_{\mu\nu}$. In the following we will take $B^\mu = 0$.

- Ψ is expected. In the Minkowski space, we have a spin zero boson associated with $\bar{a}_{\mu\nu}$ in addition to a spin one boson and a spin two boson.

- We can extract trace parts in the first two indices of $D^\alpha{}_{\mu\nu}$. This gives equivalent results.

- We can also introduce the scalar field Ψ by considering a connection of the form $\tilde{C}^\alpha_{\mu\nu} = \nabla'^\alpha q_{\mu\nu}$, where $q_{\mu\nu}$ is a symmetric tensor and isolating the trace part of $q_{\mu\nu}$.

- We now consider the case when only (Ψ, Φ) are present. $\tilde{C}^{\alpha}_{\ \mu\nu}$ is given by the following expression:

$$\tilde{C}^{\alpha}_{\ \mu\nu} = \delta^{\alpha}_{\ (\mu}\nabla'_{\ \nu)}[ln(1+\Phi)] - \frac{1}{2}g_{\mu\nu}\nabla'^{\alpha}[ln(1+\Phi)] + g_{\mu\nu}\nabla'^{\alpha}\Psi \quad (15)$$

- We obtain the following expression for the modified curvature scalar:

$$\mathcal{R} = R' - \frac{3}{2}\frac{1}{(1+\Phi)^2}(\nabla'\Phi)^2 + 3(\nabla'\Psi)^2 + \frac{3}{(1+\Phi)}[(\nabla'_{\ \kappa}\Phi)(\nabla'^{\kappa}\Psi)] \quad (16)$$

- We have the following generalization of Einstein's equation:

$$\begin{aligned}
\nabla'_{\kappa}\nabla'^{\kappa}\Phi &- \frac{1}{(1+\Phi)}(\nabla'\Phi)^2 = 0 \qquad (17)\\
\nabla'_{\kappa}\nabla'^{\kappa}\Psi &= 0\\
G'_{\mu\alpha} &= 8\pi[P'_{\mu\alpha} + \frac{3}{16\pi}P'_{\mu\alpha}(\Phi)\\
&- \frac{3}{8\pi}P'_{\mu\alpha}(\Psi) - \frac{3}{8\pi}P'_{\mu\alpha}(\Psi,\Phi)]\\
P'_{\mu\alpha}(\Psi,\Phi) &= \frac{1}{(1+\Phi)}[(\nabla'_{(\mu}\Psi)(\nabla'_{\alpha)}\Phi) - \frac{1}{2}g_{\mu\alpha}(\nabla'_{\kappa}\Psi)(\nabla'^{\kappa}\Phi)]
\end{aligned}$$

- The first two equations are the equations for Φ and Ψ respectively.

- $P'_{\mu\alpha}$ is the stress-tensor of ordinary matter fields associated with the Levi-Civita connections.
- $P'_{\mu\alpha}(\Phi)$ (for small Φ) and $P'_{\mu\alpha}(\Psi)$ are the stress-tensors of ordinary scalar fields associated with the Levi-Civita connections.
- We find that $P'_{\mu\alpha}(\Psi)$ come with an opposite sign thus contributing a negative stress-tensor to the Einstein's equations. This can be useful to dark energy search.
- $P'_{\mu\alpha}(\Phi)$ can be useful to explain inflation.
- We find that coupling of Ψ with Φ gives another contribution to source stress tensor which can be positive or negative.
- The exact equations are consistent with: $\nabla'^{\alpha} G'_{\mu\alpha} = 0$.

- We now turn to the issue of non-metricities associated with Φ, Ψ and their geometrical significance.
- We define the the non-metricity tensor by:

$$Q_{\mu\alpha\beta} = -\nabla_\mu g_{\alpha\beta} = g_{\alpha\beta} \nabla'_\mu [ln(1+\Phi)]; \;\; 2 g_{\mu(\alpha} \nabla'_{\beta)} \Psi \tag{18}$$

- We can split $Q_{\mu\alpha\beta}$ into a trace Q_μ and traceless part $\bar{Q}$ in the last two indices:

$$Q_{\mu\alpha\beta} = Q_\mu g_{\alpha\beta} + \bar{Q}_{\mu\alpha\beta}; \tag{19}$$

- We find that $\bar{Q}_{\mu\alpha\beta} = 0$, for Φ.
- In this case we can preserve the light cone under parallel transport using the reparameterization invariance of the geodesic equation: $t^{\mu}\nabla_{\mu}t^{\nu} = f(x)t^{\nu}$, where $f(x) = \frac{1}{2}t^{\mu}\nabla'_{\mu}[ln(1+\Phi)]$.
- However, this may cause problem to time-orientation when Φ is strong.
- Both Q_{μ} and $\bar{Q}_{\mu\alpha\beta}$ can be finite for Ψ. Corresponding connections do not preserve the light cone under parallel transport and we no longer have the local Minkowski structure of spacetime.
- Thus, we can not have exact $(3+1)$ -splitting of the underlying manifold into space and time. We find that departure from local Minkowski geometry can give new fields like dark matter and dark energy not found ordinarily.

- As a first approximation, we can use semiclassical and classical theories like quantum and classical fields in curved spaces to find the effects of (Ψ, Φ) when the full quantum theory is not much significant. This will be useful in cosmology. Lastly, the action for scalar-tensor theories is given by:

$$S = \int d^4x\sqrt{-g}\left[f(\phi, R') - \zeta(\phi)(\nabla'\phi)^2\right] + S_m(\psi, g_{\mu\nu}) \tag{20}$$

- Where, ψ represents other matters including radiation. We find that the fields (Ψ, Φ) can be described by such theories with $\zeta(\Psi) = -3$ and $\zeta(\Phi) = \dfrac{3}{2}\dfrac{1}{(1+\Phi)^2}$. $f(\phi, R') = R'$ for both fields. This indicates some of the quintessence, k-essence scalars can have purely geometrical origin similar to (Ψ, Φ).

- We now return to expression of scalar curvature:,

$$\mathcal{R} = R' - \frac{3}{2}(\nabla'\Omega)^2 + 3(\nabla'\Psi)^2 + 3[(\nabla'_\kappa\Omega)(\nabla'^\kappa\Psi)] \tag{21}$$

where $\Omega = ln[1+\Phi]$.

- We can add suitable (Φ, Ψ) dependent terms in the above equation including potential terms. We then have:

$$\mathcal{L} = \mathcal{R} - \mathcal{V} \tag{22}$$

- Here $\mathcal{V}$ contains the added terms which should be consistent with: $\nabla'^\alpha G'_{\mu\alpha} = 0$, if we express the equations in the form similar to Eqn.(17).
- The connections are to be evaluated using the solutions obtained from the above Lagragian density.

- In this talk, we found that it is appropriate to introduce non-metricity in quantum gravity including the free theory.
- This is in line with the gradual generalization of the geometrical structures required to discuss new observations.
- We have discussed a simple model that introduces two scalar fields towards this end. A few cosmological observations like inflation and dark energy can be related to these fields.
- This indicates that ordinary matters are related with Lorentz invariant physics while dark energy and dark matter may be associated with more general space-time geometry.
- In general, this leads us to modify the Einstein-Palatini action.

- Further details can be found in: K. Ghosh, arXiv:1910.12633v4 [gr-qc]

- Thank you.