# Well-posed initial value formulation of scalar-tensor effective field theory

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Based on

ADK and Harvey S. Reall:
*Well-posed formulation of scalar-tensor effective field theory*,
Phys. Rev. Lett. **124**, 221101
arXiv: 2003.04327 [gr-qc]

*Well-posed formulation of Lovelock and Horndeski theories*,
Phys. Rev. D **101**, 124003
arXiv: 2003.08398 [gr-qc]

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# Motivation

Precision tests of General Relativity (GR) in a strong field, highly dynamical regime

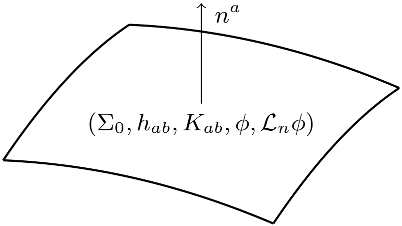
- Detection: searching for signals that matches with templates
- Numerical relativity simulations of mergers of compact objects – theoretical templates of GW signatures
- Necessary condition: well-posed initial value formulation

# Well-posedness

Given suitable initial data on a (non-characteristic) Cauchy surface $\Sigma_0$ that satisfies the constraints, the initial value problem is *well-posed* if

i) there exists a unique solution of the equations of motion,

ii) the solution depends continuously on the initial data (in a suitable norm).



# Effective field theories

EFT provides a framework to parameterize strong field deviations from GR: enumerate all higher derivative terms with the desired symmetry

Scalar-tensor theories in $4$ dimensions (with parity symmetry) [Weinberg (2008)]:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\Big(\underbrace{R - X + V(\phi)}_{\substack{2\partial \text{ theory} \\ \text{Einstein-scalar-field} \\ \text{theory}}} + \underbrace{\alpha(\phi)X^2 + \beta(\phi)\mathcal{L}_{GB}}_{4\partial \text{ terms}} + \underbrace{\ldots}_{\text{higher } \partial \text{ terms}}\Big)$$

with $X \equiv -\frac{1}{2}(\partial\phi)^2$ and

$$\mathcal{L}_{GB} = \frac{1}{4}\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} R_{\mu_3\mu_4}{}^{\nu_3\nu_4}$$

Theory with $\alpha(\phi) = 0$, $\beta(\phi) \neq 0$: Einstein-scalar-Gauss-Bonnet (EsGB) theory.

Shift invariance in the scalar field: $\beta = \lambda\phi$ and $\lambda$, $\alpha$ and $V$ are constants;

$\alpha, \lambda$ set a scale for UV physics.

# Properties of the 4-derivative theory

Equations of motion are second order (the theory is in the Horndeski class):

$$E^{\mu\nu} \equiv -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = 0 \qquad\qquad E_\phi \equiv -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta \phi} = 0$$

$$\begin{aligned} E^\mu{}_\nu \quad &= \quad G^\mu{}_\nu - \frac{1}{2}\left(2X + \alpha(\phi)X^2\right)\delta^\mu_\nu - (1+\alpha(\phi)X)\nabla^\mu\phi\nabla_\nu\phi \\ &\quad + \left(\beta''(\phi)\nabla_{\nu_1}\phi\nabla^{\mu_1}\phi + \beta'(\phi)\nabla_{\nu_1}\nabla^{\mu_1}\phi\right)\delta^{\mu\mu_1\mu_2\mu_3}_{\nu\nu_1\nu_2\nu_3}R_{\mu_2\mu_3}{}^{\nu_2\nu_3} \end{aligned}$$

$$\begin{aligned} E_\phi \quad &= \quad -(1+6\alpha(\phi)X)\Box_g\phi - 2\alpha(\phi)X\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\nabla_{\mu_1}\phi\nabla^{\nu_1}\phi\nabla_{\mu_2}\nabla^{\nu_2}\phi \\ &\quad +3\alpha'(\phi)X^2 - \frac{1}{4}\beta'(\phi)\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4}R_{\mu_1\mu_2}{}^{\nu_1\nu_2}R_{\mu_3\mu_4}{}^{\nu_3\nu_4} \end{aligned}$$

Black holes have scalar hair in EsGB gravity

# Properties of the 4-derivative theory

**Regime of validity.** Fields define a length scale $L$

$$L^{-1} = \text{max}\left\{|\mathcal{R}_{\mu\nu\rho\sigma}|^{1/2}, |\nabla_\mu\phi|, |\nabla_\mu\nabla_\nu\phi|^{1/2}\right\}$$

$L$ must be large compared to UV length scale $\Rightarrow$ *weak coupling*
$4\partial$ST terms in the e.o.m. are small compared to $2\partial$ST terms:

$$|\alpha(\phi)|,\ |\alpha'(\phi)|,\ |\beta'(\phi)|,\ |\beta''(\phi)| \ll L^2$$

The theory loses its predictive power at strong coupling:

- Analytical evidence: [Papallo & Reall (2017)]
- Numerical evidence: [Pretorius & Ripley (2019)]

> The weakly coupled assumption is compatible with strong-field phenomena!

(e.g. the dynamics of black holes that are large compared to the scale set by couplings)

# In search of a good gauge (and gauge-fixing)

Well-posed initial value formulation of diffeomorphism-invariant theories requires finding a "good" gauge and a good way of fixing the gauge

In GR, the simplest method:
harmonic gauge $H^\mu = 0$ [Choquet-Bruhat (1952)]

$$H^\mu \equiv g^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu = -g^{\nu\rho}\Gamma^\mu_{\nu\rho}[g]$$

with gauge-fixed equations

$$G^{\mu\nu} + P_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha = 0 \qquad (1)$$

where $P_\alpha{}^{\beta\mu\nu} = \delta_\alpha^{(\mu} g^{\nu)\beta} - \frac{1}{2}\delta_\alpha^\beta g^{\mu\nu}$. Equivalently,

$$R_{\mu\nu} + 2\partial_{(\mu}H_{\nu)} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \mathcal{F}_{\mu\nu}(g,\partial g) = 0$$

Nonlinear wave equation for $g_{\mu\nu}$: IVP is locally well-posed

However, IVP for harmonic gauge e.o.m. of $4\partial$ST with $\beta \neq 0$ is not well-posed, even at weak coupling [Papallo & Reall (2017)]

# Solution: modified harmonic gauge

Consider a 4d spacetime $(M, g)$ and introduce two auxiliary (inverse) Lorentzian metrics: $\tilde{g}^{\mu\nu}$ and $\hat{g}^{\mu\nu}$.
Define

$$H^\mu \equiv \tilde{g}^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu = -\tilde{g}^{\nu\rho}\Gamma^\mu_{\nu\rho}[g] \tag{2}$$

The modified harmonic gauge condition is $H^\mu = 0$.

Recall

$$E^{\mu\nu} = -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \qquad E_\phi = -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta\phi} \tag{3}$$

We now define

$$E^{\mu\nu}_{\mathrm{mhg}} = E^{\mu\nu} + \hat{P}_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha \tag{4}$$

where $\hat{P}_\alpha{}^{\beta\mu\nu} = \delta_\alpha^{(\mu}\hat{g}^{\nu)\beta} - \frac{1}{2}\delta_\alpha^\beta\hat{g}^{\mu\nu}$.
The modified harmonic gauge equations of motion are then

$$E^{\mu\nu}_{\mathrm{mhg}} = 0 \qquad E_\phi = 0 \tag{5}$$

Setting $\tilde{g}^{\mu\nu} = \hat{g}^{\mu\nu} = g^{\mu\nu}$ recovers the usual harmonic gauge equations of motion.

# Statement of the main result

## Theorem

*The modified harmonic gauge equations of motion*

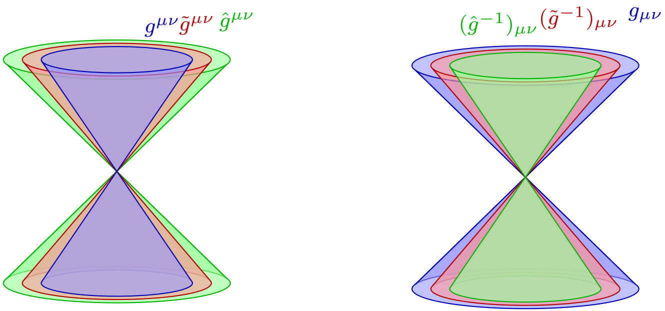
$$E^{\mu\nu}_{\text{mhg}} \equiv E^{\mu\nu} + \hat{P}_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha = 0 \qquad\qquad E_\phi = 0$$

*admit a locally well-posed initial value problem in the following two theories*

(i) *Einstein-scalar-field theory ($2\partial$ST) and*

(ii) *the weakly coupled $4$-derivative EFT ($4\partial$ST)*

*provided that the causal cones of $g$, $\tilde{g}$ and $\hat{g}$ are related as below.*



# Main idea

A sufficient condition for the nonlinear equations to admit a locally well-posed IVP is that the gauge-fixed equations of motion are *strongly hyperbolic*.

Strong hyperbolicity requires the diagonalizability of a certain matrix $M(\xi_i)$ that depends on a generic covector $\xi_i$. Its eigenvectors are related to polarizations of high frequency plane wave solutions of the gauge-fixed equations, its eigenvalues represent the speeds of propagation of these modes:

(i) "pure gauge" modes propagate along the null cone of $\tilde{g}^{\mu\nu}$

(ii) "gauge condition violating" modes propagate along the null cone of $\hat{g}^{\mu\nu}$

(iii) "physical" polarizations
- propagate along the null cone of $g^{\mu\nu}$ is Einstein-scalar-field theory
- propagate along characteristic hypersurfaces that are "almost null" in weakly coupled $4\partial$ST

In the original harmonic gauge $\tilde{g}^{\mu\nu} = \hat{g}^{\mu\nu} = g^{\mu\nu}$, the space associated with the unphysical modes (i)-(ii) is highly degenerate: $M(\xi)$ fails to be diagonalisable (generically)

# More recent results

- The modified harmonic formulation can also be used to obtain a local well-posedness in EFTs involving nonminimal couplings between gravity and a Maxwell field [Davies & Reall (2021)]

- The modified harmonic formulation has been used to perform numerical relativity simulations of black hole binaries in EsGB theories [East & Ripley (2021)].

- Alternative attempts to evolve e.o.m. in gravitational EFTs:

  - methods based on the Isreal-Stewart formulation of viscous relativistic fluids ("fixing the equations") [Cayuso, Lehner (2020)], [Gerhardinger, Giblin, Tolley, Trodden (2022)], [Lara, Bezares, Barausse (2022)], [Franchini, Bezares, Barausse & Lehner (2022)]

  - perturbation theory in couplings, e.g. [Witek, *et al.*], [Okounkova, *et al.*]; more recently [Ghersi & Stein (2021)] dynamical renormalization group approach to study binary black hole systems