When can a teleparallel geometry be classified by its scalar polynomial invariants?

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Related Publications

- A. A. Coley, R. J. van den Hoogen and DDM (2020)
  Invariants, symmetry and classification in teleparallel gravity.
  JMP, Vol 61, pp 072503.
- DDM, R. J. van den Hoogen, A. A. Coley (2021)
  Four dimensional teleparallel geometries not characterized by scalar polynomial curvature invariants,
  JMP, Vol 62, pp 052501.

Teleparallel Gravity: A frame basis perspective

A *(co)frame basis*, ($\{\mathbf{h}^a\}$) $\{\mathbf{h}_a\}$, is defined as

$$\mathbf{h}_a = h_a^{\ \alpha}(x^\alpha)\partial_{x^\alpha}\,, \quad \mathbf{h}^a = \mathbf{h}^a_{\ \alpha}(x^\alpha)dx^\alpha, \tag{1}$$

with $\alpha \in [1,4]$ and $a \in [1,4]$, such that, $h^a_{\ \alpha} h_b^{\ \beta} = \delta^a_{\ b}$.

The coframe basis must satisfy:

$$g_{\alpha\beta} = \eta_{ab} h^a_{\ \alpha} h^b_{\ \beta}, \quad \eta = diag(-1,1,1,1), \ a,b \in [1,4]. \tag{2}$$

Generally this is an *anholonomic frame* since $[\mathbf{h}_a, \mathbf{h}_b] = f^c_{\ ab}\mathbf{h}_c \neq \mathbf{0}$..

Encode information into the torsion tensor

$$\mathbf{T}(\mathbf{h}^a, \mathbf{h}^b) = \nabla_a \mathbf{h}_b - \nabla_b \mathbf{h}_a - [\mathbf{h}_a, \mathbf{h}_b] \tag{3}$$

Want: a *metric-compatible* connection that is *flat*.

Recipe for a teleparallel geometry

A twist: the coframe basis $\{h^a{}_\alpha\}$ is acted on by $SO^+(1,3)$:

$$\mathbf{h}'^a = \Lambda^a{}_b \mathbf{h}^b, \quad \Lambda^a{}_b = \Lambda^a{}_b(x^\alpha). \tag{4}$$

Two connections:
1) The *Weitzenböck* connection $(\Gamma^a{}_{cb})$

$$\nabla_b \mathbf{h}^a = \Gamma^a{}_{cb}\mathbf{h}^c = 0 \;\; (h^c{}_\gamma h_a{}^\alpha \nabla_\alpha h_b{}^\beta = h^c{}_\gamma h_a{}^\alpha (\partial_\alpha h_b{}^\beta + h_b{}^\delta \Gamma^\beta{}_{\alpha\delta}) = 0). \tag{5}$$

2) *A spin-connection* $(\omega^a{}_{bc})$

$$\omega^a{}_{bc} = \Lambda^a{}_d h_c{}^\mu (\partial_\mu \Lambda_b{}^d). \tag{6}$$

Therefore, $\nabla_a \mathbf{h}_b = \omega^c{}_{ba}\mathbf{h}_c$. and we will say a frame is **proper** if $\omega^c{}_{ba} = 0$.

The torsion tensor is the only non-zero tensor,

$$T^e{}_{cd} = \omega^e{}_{cd} - \omega^e{}_{dc} - f^e{}_{cd}. \tag{7}$$

Notation: $\nabla_d T^a{}_{bc} = T^a{}_{bc|d}$

Irreducible decomposition of torsion

Irreducible parts of torsion:

$$T_{abc} = \frac{2}{3}(t_{abc} - t_{acb}) + \frac{1}{3}(g_{ab}v_c - g_{ac}v_b) + \epsilon_{abcd}a^d, \tag{8}$$

where

$$V_a = T^b{}_{ba}, \tag{9}$$

$$A_a = \tfrac{1}{6}\epsilon_{abcd}T^{abc}, \tag{10}$$

$$t_{abc} = \tfrac{1}{2}(T_{abc} + T_{bac}) + \tfrac{1}{6}(g_{ab}V_c + g_{ac}V_b) - \tfrac{1}{3}g_{ab}V_c; \tag{11}$$

the **vector**, **axial** and **purely tensorial** parts of the torsion tensor, respectively.

$t_{(ab)c}$ has the following identities:

$$g^{ab}t_{(ab)c} = 0,\ t_{(ab)c} = t_{(ba)c},\ t_{(ab)c} + t_{(bc)a} + t_{(ca)b} = 0. \tag{12}$$

Teleparallel Equivalent of General Relativity (TEGR)

A new action with matter fields Φ^I:

$$S = \int d^4x \left(\mathcal{L}_{TG}(h^a{}_\alpha, \omega^a{}_{b\mu}) + \mathcal{L}_M(h^a{}_\alpha, \Phi^I) \right), \quad \mathcal{L}_{TG}(h^a{}_\alpha, \omega^a{}_{b\mu}) = \frac{h}{2\kappa} T. \tag{13}$$

where

$$T = \frac{1}{2} T^a{}_{bc} S_a{}^{bc}, \text{ with } S_a{}^{bc} = \frac{1}{2}\left(T_a{}^{bc} + T^{cb}{}_a - T^{bc}{}_a \right) - \delta_a{}^c V^b + \delta_a{}^b V^c. \tag{14}$$

Here, T is the *torsion scalar*, and $S_a{}^{bc}$ is the *super potential*,

Vary w.r.t. to $h^a{}_\mu$:

$$\kappa \Theta_a{}^\mu = \frac{\kappa}{h} \frac{\partial \mathcal{L}_{TG}}{\partial h^a{}_\mu} \quad = \quad \frac{1}{h} \partial_\nu \left(h S_a{}^{\mu\nu} \right) - h_a{}^\beta S_c{}^{\nu\mu} T^c{}_{\nu\beta} + \frac{1}{2} h_a{}^\mu T - \omega^c{}_{a\nu} S_c{}^{\mu\nu}. \tag{15}$$

Einstein-Hilbert plus a total divergence:

$$\mathcal{L}_{TG} = \dot{\mathcal{L}}_{GR} + \frac{1}{\kappa} \partial_\mu (h T^\mu), \quad \dot{\mathcal{L}}_{GR} = \frac{1}{\kappa} R. \tag{16}$$

Scalar Polynomial Invariants (SPIs)

Question: How can we distinguish two teleparallel geometries coming from the same analogue in GR?

With teleparallel geometries, we only have the torsion tensor and its covariant derivatives to work with:

$$T_{abc}, T_{abc|d_1}, \ldots T_{abcd|n} \tag{17}$$

Hereafter, I will refer to these as *the torsion tensors of the space*

We can take full contractions of copies of torsion tensors to produce scalars. For example,

$$T^{a}{}_{bc}T_{a}{}^{bc},\ T^{a}{}_{bc}T^{cb}{}_{a},\ \ T^{a}{}_{bc}T^{bc}{}_{a}.T^{a}{}_{bc|d}T_{a}{}^{bc|d}, \ldots \tag{18}$$

Conceptually this idea is straightforward and we could in principle consider the set of all SPIs constructed from the torsion tensors, $\mathcal{I}_T$

Notation: We will denote $\mathcal{I}_R$ as the set of all SPIs constructed from the curvature tensors.

An example from GR: VSI spacetimes

In 4D, there is a class of geometries which cannot be characterized locally using $\mathcal{I}_R$, known as *degenerate Kundt spacetimes*:

$$ds^2 = 2\boldsymbol{\ell}\mathbf{n} - \mathbf{m}\bar{\mathbf{m}}$$
$$\boldsymbol{\ell} = du, \mathbf{n} = Hdu + dv + Wd\zeta + \bar{W}d\bar{\zeta},\ \mathbf{m} = P^{-1}d\zeta,\ \bar{\mathbf{m}} = P^{-1}d\bar{\zeta}. \tag{19}$$

The real-valued functions H and P, along with the complex-valued function W satisfy:

$$H_{,vvv} = 0,\ W_{,vv} = 0,\ \text{and } P_{,v} = 0. \tag{20}$$

A special subclass where all SPIs formed from the curvature tensor are zero (Pravda, Pravdová, Coley and Milson 2002) [2] arise when:

$$\begin{aligned} H &= \frac{\epsilon^2}{4(\zeta+\bar{\zeta})^2}v^2 + H_1(u,\zeta,\bar{\zeta})v + H_0(u,\zeta,\bar{\zeta}), \quad \text{where } \epsilon = 0 \text{ or } 1, \\ W &= \frac{\epsilon}{(\zeta+\bar{\zeta})}v + W_0(u,\zeta,\bar{\zeta}),\ \text{and } P = 1. \end{aligned} \tag{21}$$

Notation We say these are VSI$_R$ geometries while their teleparallel analogues will be VSI$_T$ geometries.

Alignment classification

In a complex null (co)frame ($\{\mathbf{h}^a\} = \{\mathbf{n}, \boldsymbol{\ell}, \bar{\mathbf{m}}, \mathbf{m}\}$) $\{\mathbf{h}_a\} = \{\boldsymbol{\ell}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, a boost is

$$\boldsymbol{\ell}' = \lambda \boldsymbol{\ell}, \;\; \mathbf{n}' = \lambda^{-1}\mathbf{n}. \tag{22}$$

For any rank n tensor, $\mathbf{T}$:

$$T'_{a_1 a_2 \ldots a_n} = \lambda^{b_{a_1 a_2 \ldots a_n}} T_{a_1 a_2 \ldots a_n}, \text{ where } b_{a_1 a_2 \ldots a_n} = \textstyle\sum_{i=1}^{n} (\delta_{a_i 1} - \delta_{a_i 2}). \tag{23}$$

The **boost weight** of the frame component $T_{a_1 a_2 \ldots a_p}$ is $b_{a_1 a_2 \ldots a_n}$.

The **boost order**, $\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell})$, is the maximum b.w. of a tensor, $\mathbf{T}$ for fixed $\boldsymbol{\ell}$.

For **T**, with $\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell}) \leq 2$ for all $\boldsymbol{\ell}$ can be divided into **alignment types** if for some $\boldsymbol{\ell}$:

$$\begin{array}{cccccc} \textit{Type} & \mathbf{G} & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{N} \\ \mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell}) & 2 & 1 & 0 & -1 & -2 \end{array} \tag{24}$$

We will say $\boldsymbol{\ell}$ is $\mathbf{T}$-aligned if $\mathcal{B}_{\mathbf{T}}(\boldsymbol{\ell}) < 2$, while if **T** vanishes, then it is of alignment type **O**.

Newman-Penrose formalism

We will use the Newman-Penrose formalism for the spin connection.

This requires a complex null frame $\{\boldsymbol{\ell}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$ so that the only non-vanishing inner-products are

$$\ell_a n^a = -m_a \bar{m}^a = 1.$$

The complex spin connection components can be labelled as:

$$\begin{aligned}
&\textbf{b.w. 2}: -\kappa = \omega_{311}, \\
&\textbf{b.w. 1}: -\rho = \omega_{314},\ -\sigma = \omega_{313}, -\epsilon = \frac{1}{2}(\omega_{211} - \omega_{431}), \\
&\textbf{b.w. 0}: -\tau = \omega_{312}, \pi = \omega_{421}, \alpha = \frac{1}{2}(\omega_{124} - \omega_{344}), -\beta = \frac{1}{2}(\omega_{213} - \omega_{433}), \\
&\textbf{b.w. -1}: \mu = \omega_{423},\ \lambda = \omega_{424}, \gamma = \frac{1}{2}(\omega_{122} - \omega_{342}), \\
&\textbf{b.w. -1}: \nu = \omega_{422}.
\end{aligned} \tag{25}$$

With this classification approach we have corollary coming from (Hervik 2011) [1]:

Corollary

A teleparallel geometry is $\mathcal{I}_T$-degenerate if and only if the torsion tensors are of alignment type **II** *or more special relative to some common null coframe.*

In terms of the irreducible parts of the torsion tensor,

Lemma

A torsion tensor is of alignment type **II** *or more special if and only if relative to a common null frame:*

- *The purely tensor-part of the torsion tensor, $t_{(ab)c}$, is of alignment type* **II**, **III**, **N** *or* **O**.
- *The vector and axial parts of torsion are of alignment type* **II**, **III** *or* **O**.

ℓ-proper frames

We can always choose to work in a proper frame, where all of these coefficients vanish.

Starting from a proper frame, we can apply a null rotation about **n** to align $\boldsymbol{\ell}$:

$$\mathbf{n}' = \mathbf{n}, \boldsymbol{\ell}' = \boldsymbol{\ell} + \bar{E}\mathbf{m} + E\bar{\mathbf{m}} + |E|^2\mathbf{n}, \mathbf{m}' = \mathbf{m} + E\mathbf{n} \tag{26}$$

so that the only non-zero spin-coefficients are

$$\begin{aligned}
\kappa' &= |E|^2\tau + \Delta E + E\delta\bar{E} + \bar{E}\bar{\delta}E + |E|^2 DE, \\
\sigma' &= E\tau + EDE + \bar{\delta}E, \\
\rho' &= \bar{E}\tau + \bar{E}DE + \delta E, \\
\tau' &= \tau + DE.
\end{aligned} \tag{27}$$

This frame is an ℓ-*proper frame* and in the case of $\mathcal{I}_T$-degeneracy it is adapted to the torsion tensors.

Kundt coframes: A frame anzatz

Theorem

If a teleparallel geometry admits a coframe, $\{\mathbf{n}, \ell, \bar{\mathbf{m}}, \mathbf{m}\}$, where the torsion tensor is of alignment type **II** *and*

$$\kappa = \sigma = \rho = 0, \tag{28}$$

then coordinates can be chosen so that the coframe is of the form:

$$\begin{aligned} \ell &= du \\ \mathbf{n} &= dv + H(u,v,x^1,x^2)du + W_i(u,v,x^1,x^2)dx^i \\ \mathbf{m} &= M_0(u,v,x^1,x^2)du + M_1(u,x^1,x^2)(dx^1 + idx^2) \end{aligned} \tag{29}$$

where H and W_i are real-valued functions and M_0 and M_1 are complex-valued functions.

We will call such a coframe, a **Kundt coframe**.

$\mathcal{I}_T$-degenerate teleparallel geometries

Theorem

For any $\mathcal{I}_T$-degenerate teleparallel geometry, coordinates can be chosen so that the Kundt coframe takes the form:

$$\begin{aligned} \boldsymbol{\ell} &= du \\ \mathbf{n} &= dv + H(u,v,x^1,x^2)du + W_i(u,v,x^1,x^2)dx^i \\ \mathbf{m} &= M_0(u,v,x^1,x^2)du + M_1(u,x^1,x^2)(dx^1 + idx^2) \end{aligned} \tag{30}$$

where M_0 and M_1 are complex-valued functions, H and W_i are real-valued functions and M_0, H and W_i are polynomial in the v-coordinate:

$$M_0 = M_0^{(1)}v + M_0^{(0)},\ H = H^{(2)}v^2 + H^{(1)}v + H^{(0)} \text{ and } W_i = W_i^{(1)}v + W_i^{(0)}. \tag{31}$$

Relative to this coframe, the following spin connection components must vanish

$$\kappa = \rho = \sigma = \epsilon = 0 \tag{32}$$

and an ℓ-preserving Lorentz transformation can be used to set all other spin connection components to zero except $\tau = \tau(u,x,y)$.

A proper frame formulation: Defining the metric functions

Lemma

For any $\mathcal{I}_T$-degenerate teleparallel geometry, the Kundt coframe in equations (30) *and* (31) *is related to a proper coframe by a null rotation about* **n** *with the frame functions,* W_i *and* M_0 *below and null rotation parameter satisfy the following constraints:*

$$E \quad = E^{(1)}(u,x^1,x^2)v + E^{(0)}(u,x^1,x^2). \tag{33}$$

with $\Delta E, = \bar{\delta} E = \delta E. = 0.$

Theorem

For any $\mathcal{I}_T$-degenerate teleparallel geometry with an ℓ-proper Kundt coframe of the form (30) *and* (31) *with* $\tau \neq 0$*, the null rotation parameter* $E(u,x^i)$ *is real-valued and the frame functions take the following form:*

$$H = (ln\ E^{(1)})_{,u} v + \frac{E^{(0)}_{,u}}{E^{(1)}}, \text{ and } W_i = (\ln E^{(1)})_{,x^i} v + \frac{E^{(0)}_{,x^i}}{E^{(1)}}. \tag{34}$$

VSI teleparallel geometries

Theorem

The class of VSI_T teleparallel geometries are given by the coframe:

$$\begin{aligned} \boldsymbol{\ell} &= du \\ \mathbf{n} &= dv + H(u,v,x^1,x^2)du + W_i(u,x^1,x^2)dx^i \\ \mathbf{m} &= M_0(u,x^1,x^2)du + M_1(u)(dx^1 + idx^2) \end{aligned} \tag{35}$$

where M_0 and M_i are complex-valued functions, H and W_i are real-valued functions and H is linear in the v-coordinate:

$$H = H^{(1)}(u,x^1,x^2)v + H^{(0)}(u,x^1,x^2). \tag{36}$$

This coframe is proper, as the spin connection must be zero in this frame.

References

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