Semiclassical theories of gravity and their linear stability

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Motivations

- **Quantum matter - gravity interplay** is fully described only within:

  quantum gravity.

- However, under certain conditions, it is possible to analyse this interplay within semiclassical approximation:
  - QFT on curved spacetime
  - Backreaction

$$G_{ab} = 8\pi G \langle T_{ab} \rangle_\omega$$

- Effects
  - In cosmology (particle creation, some models of inflation)
  - Black Hole Physics (Hawking radiation, evaporation)
- It equates **classical** quantities with **probabilistic** ones.
- Existence and uniqueness of solutions of **SEE established** only in few cases.

Aims

- **Extend** existence theorems a bit.
- **Well posedness** of the dynamical problem is a necessary prerequisite to the analysis of numerical solutions.
- Discuss the **stability** of linear perturbations in some cases.

Plan of the talk

- Quantum fields on cosmological / curved spacetimes.
- Local existence of solutions of semiclassical gravity in cosmology.
- A simple semiclassical model.
- Stability of linear perturbations.

This talk is based on

- P. Meda, NP, (https://arxiv.org/abs/2201.10288).
- P. Meda, NP, D. Siemssen, Ann. Henri Poincaré **22** 3965-4015 (2021).
- NP, D. Siemssen, Comm. Math. Phys. **334** 171-191 (2015).
- NP, Comm. Math. Phys. **305** 563-604 (2011).

Semiclassical Einstein equation in cosmology

- Cosmological spacetimes

$$(M, g)\ , \qquad M = I \times \Sigma\ .$$

- Flat cosmological spacetimes

$$g = -dt \otimes dt + a(t)^2 dx^i \otimes dx^i\ ,$$

  - t the **cosmological time**.
  - a is the **scale factor**.
  - $H = \frac{d}{dt}\log(a)$ is the **Hubble parameter**.
  - $d\tau = a^{-1}dt$ is the **conformal time** $g = a(\tau)^2\left[-d\tau^2 \otimes d\tau + dx^i \otimes dx^i\right]\ ,$
- One DOF hence: simpler set of equation

$$\begin{cases} \nabla_a \langle T^{ab} \rangle = 0 \\ -R = 8\pi G \langle T \rangle_\omega \\ G_{00} = 8\pi G \langle T_{00} \rangle_\omega \qquad \text{at} \qquad \tau = \tau_0 \end{cases}$$

- We look for local existence and uniqueness of solutions of that system.

Quantum scalar field on curved backgrounds

- Let (M, g) be a globally hyperbolic spacetime.
- Massive scalar quantum field coupled to gravity.

$$K\varphi = -\Box\varphi + \xi R\varphi + m^2\varphi = 0$$

- The quantization is well under control.
- Assign to every spacetime a $*-$algebra of observables.

$$M \mapsto \mathcal{A}(M)$$

- $\mathcal{A}(M)$ generated by linear fields $\varphi(f)$, $f \in C_0^\infty(M)$ implementing:

$$\varphi^*(f) = \varphi(\overline{f}) \ , \qquad \varphi(Kf) = 0 \ , \qquad [\varphi(f), \varphi(h)] = i\Delta(f, h).$$

- Where $\Delta(f, h) = \Delta_R(f, h) - \Delta_A(f, h)$ (ret. minus adv. fund. solutions of $K\varphi = 0$)
- **Covariance:** This quantization procedure is coherent wrt imbeddings. *[Brunetti Fredenhagen Verch, Hollands Wald]*
- A **state** ω is a positive normalized linear functional over $\omega : \mathcal{A} \to \mathbb{C}$.
- Once a state is chosen by **GNS theorem** we can represent $\mathcal{A}(M)$ as operators over some Hilbert space $\mathfrak{H}$ and ω as a normalized vector in $\mathfrak{H}$.

Extended algebra of Wick polynomials

- $\omega : \mathcal{A} \to \mathbb{C}$ (positive, normalized, linear functional)
- Characterised by $n-$point functions $\omega_n \in \mathcal{D}'(M^n)$

$$\omega_n(f_1, \dots, f_n) := \omega(\varphi(f_1)\dots\varphi(f_n)).$$

- We need to **extend** $\mathcal{A}(M)$ **to include** objects like

$$\varphi^n(f), \qquad T_{ab}(f).$$

However, these are divergent quantities

$$\omega(\varphi^2(x)) = \lim_{y\to x} \omega_2(y, x) = \infty.$$

- We need a regularization prescription which implements some **normal ordering**.
- In a **Hadamard state** the local singular structure is universal.
Hadamard two-point function:

$$\omega_2 = \mathcal{H} + W, \qquad \mathcal{H} = \frac{U}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{\lambda^2}\right)$$

- Point splitting regularization

$$:\varphi^2:_{\mathcal{H}}(x) = \lim_{y\to x}\left[\varphi(x)\varphi(y) - \mathcal{H}(x, y)\right].$$

- There is a **regularization freedom**, fully classified by *[Hollands Wald]*
- We extend $\mathcal{A}(M)$ to include normal ordered Wick powers.

Point splitting regularization - backreaction

- **Stress-Energy Tensor:**

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}(\nabla_c\varphi\nabla^c\varphi + m^2\varphi^2) + \xi\left(G_{ab}\varphi^2 - \nabla_a\nabla_b\varphi^2 + g_{ab}\nabla_c\nabla^c\varphi^2\right).$$

  Hence, classically

$$T_{ab}(x) = \lim_{y\to x} D_{ab}\varphi(x)\varphi(y)$$

- **Expectation values:** obtained subtracting the Hadamard singularity $\mathcal{H}$ from ω_2

$$\langle :T_{ab}: \rangle_\omega := \omega(:T_{ab}:_{\mathcal{H}}) = \lim_{y\to x} D_{ab}\left[\omega_2(x,y) - \mathcal{H}(x,y)\right] + ren.freedom.$$

- The renormalization freedom is classified *[Hollands Wald]*
- It is fixed in part by the **requirement** $\nabla^a\langle :T_{ab}: \rangle = 0$. *[Moretti, Hollands Wald]*
- The price to pay is an **anomalous contribution** to the trace.

Components of $\langle T \rangle$

- The trace of the stress tensor T can be decomposed in the following three contributions

$$\langle :T: \rangle_\omega = T_{ren.freedom} + T_{anomaly} + T_{state}$$

- **Renormalization freedom**

$$T_{ren.freedom} = \beta_1 m^4 + \beta_2 m^2 R + \beta_3 \Box R$$

  - β_1 expresses a renorm. of the cosmological constant $\Longrightarrow$ adding $\beta_1 m^4$ to $\mathcal{L}$
  - β_2 expresses a renorm. of the Newton constant $\Longrightarrow$ adding $\beta_2 m^2 R$ to $\mathcal{L}$
  - β_3 is a pure quantum freedom $\Longrightarrow$ adding $\beta_3 R^2$ or $\beta_3 R_{ij} R^{ij}$ to $\mathcal{L}$
- **Anomalous term** (which cannot be reabsorbed in ren. freedom)

$$T_{anomaly} = \frac{1}{4\pi^2} \left(\frac{(6\xi - 1)^2 R^2}{288} + \frac{R_{abcd} R^{abcd} - R_{ab} R^{ab}}{720} \right)$$

- **State dependent contribution**

$$T_{state} = \left(3 \left(\xi - \frac{1}{6} \right) \Box - m^2 \right) \langle :\phi^2: \rangle_\omega \ .$$

$\langle :\phi^2: \rangle_\omega = \lim_{y \to x} (\omega_2(x, y) - \mathcal{H}(x, y)) = W(x, x).$

State dependent contributions

- Some hypotheses (state as close as possible to a "vacuum")
  - Gaussian (quasi-free) (only ω_2 matters)
  - pure, homogeneous, isotropic
- The **pure**, **homogeneous** and **isotropic** Gaussian state

$$\omega_2(x,y) = \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\overline{\zeta}_k(\tau_x)}{a(\tau_x)} \frac{\zeta_k(\tau_y)}{a(\tau_y)} \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} \mathrm{e}^{-\varepsilon k} \mathrm{d}\vec{k}$$

where $k \doteq |\vec{k}|$ and where the *temporal modes* ζ_k fulfil the equation

$$\zeta_k''(\tau) + \Omega_k^2(\tau)\zeta_k(\tau) = 0, \qquad \Omega_k^2(\tau) \doteq k^2 + a^2 m^2 + \left(\xi - \frac{1}{6}\right) R a^2$$

and satisfy the normalization condition

$$\zeta_k' \overline{\zeta}_k - \zeta_k \overline{\zeta}_k' = \mathrm{i}.$$

- ζ_k needs to be chosen in such a way that the state is **regular enough** to give finite expectation values for

$$\langle :\phi^2: \rangle_\omega, \qquad \text{and} \qquad \langle : T_{00} : \rangle_\omega$$

- There is still a freedom in the choice of ζ_k. ▸ Details.

Semiclassical equation in cosmology

- The **scale factor** a is the unique degree of freedom of the problem.
- The state we are considering is quasifree pure homogeneous, it is characterized by some **initial conditions** for the modes ζ_k.
- Due to the large symmetry the Einstein equation simplifies:

$$\begin{cases} \nabla^a \langle :T_{ab}: \rangle_\omega = 0 \\ -R(a, a'') + 4\Lambda = 8\pi G \langle :T: \rangle_\omega \left(a, a', a'', a^{(3)}, a^{(4)} \right) \\ G_{00}(\tau_0) - a^2 \Lambda = 8\pi G \langle :T_{00}: \rangle_\omega \left(a_0, a_0', a_0'', a_0^{(3)} \right), \qquad \text{at} \qquad \tau = \tau_0. \end{cases}$$

equipped with some initial conditions for a and for ζ_k.

Observations

- The first equation is **satisfied** because of the chosen **regularization** procedure.
- The **trace** of the stress tensor contains **forth order** derivatives of a.

Results

- The third equation fixes the modes ζ_k for given $(a_0, a_0', a_0'', a_0^{(3)})$.
- We rewrite the second equation to get an initial value problem which admits a unique solution $a(\tau)$ for **conformal time** $\tau \in [\tau_0, \tau_1]$ once we fix $(a_0, a_0', a_0'', a_0^{(3)})$.

A simple semiclassical problem

- Consider a **quantum field** ϕ_I propagating on Minkowski spacetime

$$\Box\phi_I - m^2\phi_I - \lambda\psi\phi_I = 0,$$

- ψ is a classical **external potential** which is constrained by a semiclassical equation

$$\psi = \Lambda + \omega(:\phi_I^2:)$$

where ω is a quantum (quasi-free) state fixed with some **initial conditions** at $t = 0$.

- ψ and ω can be constructed **recursively**,
- We use perturbation theory to evaluate $\lambda\omega(:\phi_I^2:)$, starting with the vacuum state and with the interaction Lagrangian

$$\mathcal{L}_I = -\frac{\lambda}{2}\psi:\phi^2:, \qquad V = -\frac{\lambda}{2}\int \psi:\phi^2:g d^4x$$

- The expectation value of $\omega(:\phi_I^2:)$ obtained with perturbation theory

$$:\phi_I^2: = R_V(:\phi^2:) = :\phi^2: + iT(V,\phi^2) - iV:\phi^2: + \dots$$

- fixing the vacuum state as the reference state for the free theory we have

$$\omega_I(:\phi_I^2:)(x) = \omega(R_V(:\phi^2:))(x) = -i\int d^4y\left(\Delta_F^2(y,x) - \Delta_+^2(y,x)\right)\psi(y)g(y) + \dots$$

- Truncating at first order we get an integro-differential equation for ψ

$$\psi(x) = \Lambda(x) - i\lambda \int d^4y \left(\Delta_F^2(y,x) - \Delta_+^2(y,x)\right) \psi(y) g(y)$$

$$\psi = \Lambda + \mathcal{T}(\psi)$$

- This is a fixed point equation, starting from ψ_0 one constructs

$$\psi_n = \Lambda + \mathcal{T}(\psi_{n-1}), \qquad \text{and one takes} \qquad \lim_{n\to\infty} \psi_n = \dots$$

however, we have a problem actually:

  - $\mathcal{T}$ is essentially $c \log(\Delta_{R,2m})$.
  - $\mathcal{T}$ looses derivatives

$$\|\mathcal{T}(\psi)\|_\infty \leq C \left(\|\psi\|_\infty + \|\partial\psi\|_\infty\right)$$

  - Usually, in this case, the recursive procedure does not converge even on small intervals of time

- We find an inversion formula for $a = \mathcal{T}(b)$

$$\psi = \psi_0 + \mathcal{T}^{-1}(\psi - \Lambda)$$

and $\mathcal{T}^{-1}$ has nicer properties (continuous).

The new equation can be treated by fixed point methods.

(Banach fixed point theorem)

Linearized perturbations and stability

Simple semiclassical model revisited

$$\begin{cases} \Box\phi - m^2\phi = \lambda\psi\phi, \\ g_2\Box\psi - g_1\psi = \lambda_1\langle:\phi^2:\rangle_\omega - \lambda_2\Box\langle:\phi^2:\rangle_\omega, \end{cases}$$

consider linear perturbations

$$\psi = \psi_0 + \psi_1$$

At linear order in V we have

$$g_2\Box\psi_1 - g_1\psi_1 = (\lambda_1 - \lambda_2\Box)\,\langle:\phi^2:\rangle_\omega^{(\text{lin})}, \qquad (\star)$$

where the contributions higher then the linear one are not displayed. As before,

$$\langle:\phi^2:(x)\rangle_\omega^{(\text{lin})} = -i\hbar\lambda\int_{\mathcal{M}}\left(\Delta^2_{F,\omega}(y,x) - \Delta^2_{+,\omega}(y,x)\right)\psi_1(y)dy,$$

Equation $(\star)$ is the only equation we have at linear order, and it must be seen as a dynamical equation for the linear perturbation ψ_1.

$\langle :\phi^2: \rangle_{\omega}^{(\mathrm{lin})}$

By direct insepction

$$\langle :\phi^2: \rangle_{\omega}^{(\mathrm{lin})} = -i\hbar\lambda \int_{\mathcal{M}} \left(\Delta^2_{F,\omega}(y,x) - \Delta^2_{+,\omega}(y,x) \right) \psi_1(y) dy,$$
$$= \hbar\lambda \mathcal{K}_a(\psi_1)$$

Chose Minkowski with the vacuum as background.

In that case we have explicit computations (Källen-Lehmann spectral representations), which yield

$$\mathcal{K}_a(x) := -i\left(\Delta^2_F(x) - \Delta^2_+(x)\right) = (\Box + a)\int_{4m^2}^{\infty} dM^2 \sqrt{1 - \frac{4m^2}{M^2}} \frac{1}{M^2 + a} \Delta_A(x, M^2),$$

We know everything about this function, $\hat{\mathcal{K}}_a(p_0, \vec{p})$ is fully under control ▸ Details.
The equation for **linear perturbations**

$$(g_2\Box - g_1)\psi_1(x) = (\lambda_1 - \lambda_2\Box)\langle :\phi^2: \rangle_0^{(\mathrm{lin})}(x). \qquad (\star)$$

can be written as

$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1)(x) + (g_2\Box - g_1)\psi_1(x) = 0$$

Linearized equations with a source

Consider the equation $(\star)$ sourced by $f \in C_0^\infty(\mathcal{M})$ in the form

$$\begin{aligned} \hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1)(x) + (g_2\Box - g_1)\psi_1(x) &= f(x) \\ S(\psi_1)(x) &= f(x) \end{aligned} \qquad (\star\star)$$

it respects causality: $\psi_1(x)$ depends only on what happens at $J^-(x)$.

Taking the (formal) Fourier transform of the inhomogeneous equation

$$S(-(p_0 - i0^+)^2 + |\vec{p}|^2)\hat{\psi}_1(p_0, \vec{p}) = \hat{f}(p_0, \vec{p}),$$

where

$$S(p^2) := -\left((\lambda_1 + \lambda_2 p^2)\frac{\lambda\hbar}{16\pi^2}\hat{\mathcal{K}}_a(p_0, \vec{p}) + (g_2 p^2 + g_1)\right). \qquad p^2 = -p_0^2 + |\vec{p}|^2$$

Hence, at least formally

$$\hat{\psi}_1(p_0, \vec{p}) = \frac{1}{S(-(p_0 - i0^+)^2 + |\vec{p}|^2)}\hat{f}(p_0, \vec{p})$$

To write it in position space we have to analyze

$$\mathcal{Z} := \{z | S(z) = 0\}$$

and the points where $S(z)$ has cuts.

Linear perturbation with a source

Consider the semiclassical equation with a source term $f \in C_0^\infty(\mathcal{M})$ in the form

$$\hbar\lambda(\lambda_2\Box - \lambda_1)\mathcal{K}_a(\psi_1)(x) + (g_2\Box - g_1)\psi_1(x) = f(x). \qquad (\star\star)$$

If

a) $g_2\lambda_1 - \lambda_2 g_1 \geq 0$

b) $-4m^2 < a < 0$

c) $\mathcal{Z} = \{z | S(z) = 0\}$ contains only real negative elements (suff. cond. $\lambda_2 g_2 > 0$)

we find a retarded fundamental solution

$$D_R(x) = -\sum_{s\in\mathcal{Z}} \frac{1}{S'(s)}\Delta_R(x,-s) - \frac{\lambda\hbar}{16\pi^2}\int_{4m^2}^{\infty}\sqrt{1-\frac{4m^2}{M^2}}\,\frac{(\lambda_2 M^2 - \lambda_1)}{|S(-M)|^2}\Delta_R(x,M^2)dM^2,$$

and

$$D_R : C_0^\infty \to C^\infty, \qquad \psi_1 = D_R(f).$$

ψ_1 is a solution of $(\star\star)$ and $\psi_1(t,\vec{x})$ **decays** as $1/t^{3/2}$ for large t.

Remark

A past compact solution $\psi_1 = D_R(f)$ of $(\star\star)$ can be decomposed into two parts,

$$\psi_1(x) = \psi_1^O(x) + \psi_1^C(x),$$

where $\psi_1^O = -\sum_{s\in\mathcal{Z}} \frac{1}{S'(s)} \Delta_R(f, -s)$.

- It is **not** possible to determine ψ_1^C with a finite number of initial conditions, because the integration of M^2 is over uncountably many points.
- The **homogeneous equation** $(\star)$ gives origin to a well-posed initial value problem.
- Because, the contribution due to ψ_1^C does not enter the construction of the solutions of the homogeneous equation on the whole space.
- The reason is that the Kernel of the multiplicative operator T, which acts on $\mathcal{S}(\mathbb{R}^4)$ and is defined as

  $$T(z) \doteq \frac{S(z)}{\prod_{s\in\mathcal{S}}(z-s)},$$

  contains only 0, with $z = -(p_0 - i\epsilon)^2 + |\vec{p}|^2$.
- Therefore, only the contributions due to ψ^O can give origin to non trivial solutions of the homogeneous equation $(\star)$ written as $S(z)\psi_1 = 0$.

Homogeneous equation

Consider the semiclassical equation $(\star)$ which we recall here

$$(g_2\Box - g_1)\psi_1(x) = (\lambda_1 - \lambda_2\Box)\langle\phi^2\rangle_0^{(\mathrm{lin})}(x). \qquad (\star)$$

If

a) $g_2\lambda_1 - \lambda_2 g_1 \geq 0$

b) $-4m^2 < a < 0$

c) $\mathcal{Z} = \{z | S(z) = 0\}$ contains only real negative elements (suff. cond. $\lambda_2 g_2 > 0$)

Let $\psi_1(t, \vec{x})$ be a smooth solution of eq $(\star)$ with spatial compact support. Its spatial Fourier transform is of the form

$$\tilde{\psi}_1(t, \vec{p}) = \sum_{s \in \mathcal{Z}} \left(C_+^s(\vec{p}) e^{+it\sqrt{|\vec{p}|^2 - s}} + C_-^s(\vec{p}) e^{-it\sqrt{|\vec{p}|^2 - s}} \right),$$

- The coefficients $C_\pm^s(\vec{p})$ can be determined by a suitable number of initial conditions $2|\mathcal{Z}|$.
- Furthermore, in this case, $\psi_1(t, \vec{x})$ **decays** for large time at least as $1/t^{3/2}$.

There are choices of the parameters for which the conditions written above are satisfied.

The analogy to the cosmological case

There are (not so special) choices of the parameters which make the game.

$$-R + 4\Lambda = 8\pi G \langle T \rangle_\omega .$$

$$\langle T \rangle_\omega = \left(3\left(\xi - \frac{1}{6}\right)\Box - m^2\right)\langle\phi^2\rangle_\omega + T_{anomaly} + \alpha_1 m^4 - \alpha_2 m^2 R + \alpha_3 \Box R,$$

$$T_{anomaly} = \frac{1}{2880\pi^2}\left(C^{abcd}C_{abcd} + R^{ab}R_{ab} - \frac{1}{3}R^2\right),$$

Consider as background solution the Minkowski spacetime.

- $T_{anomaly}$ is of second order, it can be discarded at linear order.
- A formal correspondence between the linearization of the traced semiclassical equation and $(\star)$ $\quad R \quad \Longleftrightarrow \quad \psi_1$

Relation between the parameters

$$g_1 = -\frac{1}{8\pi G}, \qquad g_2 = \alpha_3, \qquad \lambda = \xi, \qquad \lambda_1 = m^2, \qquad \lambda_2 = 3\left(\xi - \frac{1}{6}\right).$$

Condition a) is simply

$$\alpha_3 \frac{m^2}{m_P^2} \geq -\frac{3}{8\pi}\left(\xi - \frac{1}{6}\right), \qquad \alpha_3 \in \mathbb{R}.$$

Condition b) is $0 > a > -4m^2$.

By choosing $\xi > 1/6$, $\alpha_3 > 0$, we get the validity of condition c)

Final comments

Summary

- Banach fixed point theorem gives existence and uniqueness for short intervals of time
- Local existence of solution is thus established
- We have seen that it is necessary to rewrite the equation to get this result
- Linear stability holds for certain values of the renormalization parameters

Thanks a lot for your attention!

Constraint on the state

Explicit form

$$\langle :\phi^2: \rangle_\omega = \frac{1}{(2\pi)^3 a^2} \int_{\mathbb{R}^3} \left(|\zeta_k|^2 - C^{\mathcal{H}}_{\phi^2}(\tau, k) \right) \mathrm{d}\vec{k} + \frac{w(\tau)^2}{8\pi^2 a^2} \log\left(\frac{w(\tau_0)}{a(\tau)}\right) - \frac{w(\tau_0)^2}{16\pi^2 a^2} + \alpha_1 m^2 + \alpha_2 R$$

$$\langle :\varrho: \rangle_\omega = \frac{1}{(2\pi)^3 a^4} \int_{\mathbb{R}^3} \left(\frac{|\zeta_k'|^2}{2} + \left(k^2 + a^2 m^2 - (6\xi - 1)\, a^2 H^2\right) \frac{|\zeta_k|^2}{2} + aH\,(6\xi - 1)\, 2\mathrm{Re}(\overline{\zeta}_k \zeta_k') - C^{\mathcal{H}}_{\varrho}(\tau, k) \right) \mathrm{d}\vec{k}$$

$$- \frac{H^4}{960\pi^2} + \left(\xi - \frac{1}{6}\right)^2 \frac{3H^2 R}{8\pi^2} + \tilde{\beta}_1 m^4 - \tilde{\beta}_2 m^2 G_0{}^0 + (\tilde{\beta}_3 - \frac{\tilde{\beta}_4}{3}) I_0{}^0$$

where $w(\tau) = \Omega_0(\tau)$

$$C^{\mathcal{H}}_{\phi^2}(\tau, k) \doteq \frac{1}{2k_0} - \frac{V(\tau)}{4k_0^3}, \qquad k_0^2 \doteq \Omega_k^2(\tau = \tau_0)$$

$$C^{\mathcal{H}}_{\varrho}(\tau, k) \doteq \frac{k}{2} + \frac{a^2 m^2 - a^2 H^2 (6\xi - 1)}{4k} - \frac{a^4 m^4 + 12\left(\xi - \frac{1}{6}\right) m^2 a^4 H^2 + a^4 \left(\xi - \frac{1}{6}\right)^2 2 I_0{}^0(\tau)}{16k(k^2 + \frac{a^2}{\lambda^2})}$$

Regularity condition

$$|\zeta_k^2(\tau_0)| - C^{\mathcal{H}}_{\phi^2}(\tau_0, k) \in L^1(k^2 dk) \qquad \frac{d}{d\tau}\left[|\zeta_k^2|(\tau) - C^{\mathcal{H}}_{\phi^2}(\tau, k) \right]\Big|_{\tau=\tau_0} \in L^1(k^2\,\mathrm{d}k)$$

$$\left(\frac{|\zeta_k'|^2}{2} + \left(k^2 + a^2 m^2 - (6\xi - 1)\, a^2 H^2\right) \frac{|\zeta_k|^2}{2} + aH\,(6\xi - 1)\, 2\mathrm{Re}(\overline{\zeta}_k \zeta_k') - C^{\mathcal{H}}_{\varrho}(\tau, k) \right)\Big|_{\tau=\tau_0} \in L^1(k^2\,\mathrm{d}k).$$

Proposition

Back

Let $\psi_1 \in \mathcal{S}(\mathcal{M})$ be a Schwartz function on $\mathcal{M}$, and let $\hat{\psi}_1$ be its Fourier transform. Then

$$\mathcal{F}\left\{\langle\phi^2\rangle_0^{(\text{lin})}\right\}(p_0, \vec{p}) = \lim_{\epsilon\to 0^+} \frac{\lambda\hbar}{16\pi^2} F_a(-(p_0 - i\epsilon)^2 + |\vec{p}|^2)\hat{\psi}_1(p_0, \vec{p}),$$

given for strictly positive mass $m > 0$ and for $a > -4m^2$. The function $F_a(z)$ admits the following integral representation:

$$F_a(z) = \int_{4m^2}^{\infty} \sqrt{1 - \frac{4m^2}{M^2}} \left(\frac{1}{M^2 + a} - \frac{1}{M^2 + z}\right) dM^2,$$

and it has the following properties:

a) *$F_a(z)$ is an analytic for $z \in \mathbb{C} \setminus (-\infty, -4m^2]$ and continuous for $z = -4m^2$;*

b) *the domain $F_a(z)$ has a branch cut on $z \in (-\infty, -4m^2)$ because there the imaginary part is discontinuous (the real part is continuous but not differentiable);*

c) *$F_a(a) = 0$;*

d) *$F_a(s)$ is real for $s \in \left[-4m^2, \infty\right)$, it is strictly increasing for $s \in \left[-4m^2, \infty\right)$, and it diverges for large $|s|$;*

e) *The imaginary part of F_a admits the following integral representation:*

$$\text{Im}(F_a(z)) = \text{Im}(z) \int_{4m^2}^{\infty} \sqrt{1 - \frac{4m^2}{M^2}} \left(\frac{1}{|M^2 + z|^2}\right) M^2,$$

it is strictly positive for $\text{Im}(z) > 0$, and strictly negative for $\text{Im}(z) < 0$. Furthermore, it vanishes for $z \in (-4m^2, \infty)$, and it is discontinuous on $z \in (-\infty, -4m^2)$ (the absolute value is finite).

Finally, for $z \notin (-\infty, 0)$ and $a > 0$, $F_a(z)$ takes the form

$$F_a(z) = 2\sqrt{\frac{z + 4m^2}{z}} \log\left(\frac{\sqrt{z + 4m^2} + \sqrt{z}}{2m}\right) - 2\sqrt{\frac{a + 4m^2}{a}} \log\left(\frac{\sqrt{a + 4m^2} + \sqrt{a}}{2m}\right).$$