GR23 @ Beijing, 2022/7/6

A novel formulation for the evolution of relativistic rotating stars

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Based on arXiv[gr-qc]: 2204.09943

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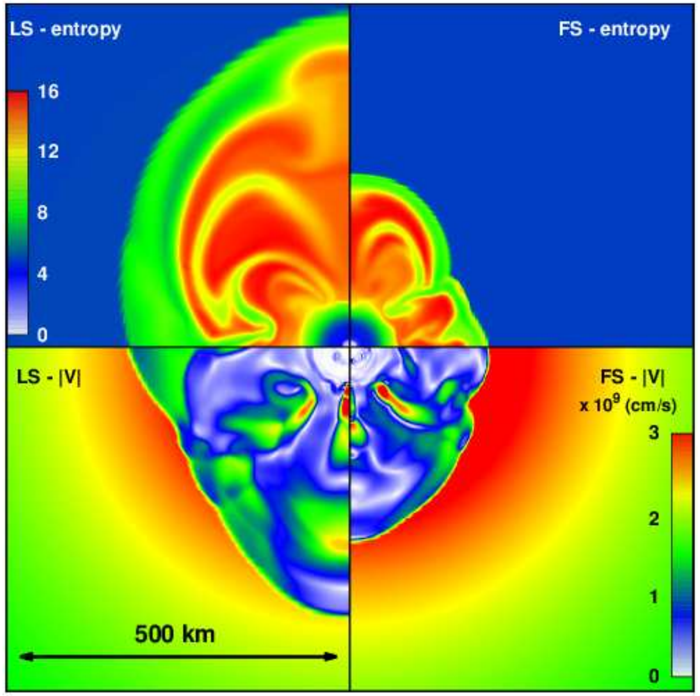
Supernova explosion

Supernova — PNS — NS

- Core-collapse supernovae($M \geq 8M_{sun}$)
  - Many References...
    (Marek&Janka '09, Müller+ '12, Bruenn+ '13,'16 Takiwaki+ '14, Dolence+ '15, Lentz+ '15', Melson+ '15, O'Connor&Cough '15, Just+ '15, Kuroda+ '16, Summa+ '16, Pan+ '16, Roberts+ '16, Burrows+ '16, Andresen+ '17, $\cdots$)
  - **Neutrino Transport:**
    First-principle calculation by Boltzmann equation
  - Successfull explosion:
    Progenitor, EOS, Rotation, Dimension, GR?
    Harada+ '19&'20, Akaho+ '21, Iwakami+ '21, Akaho+ '22



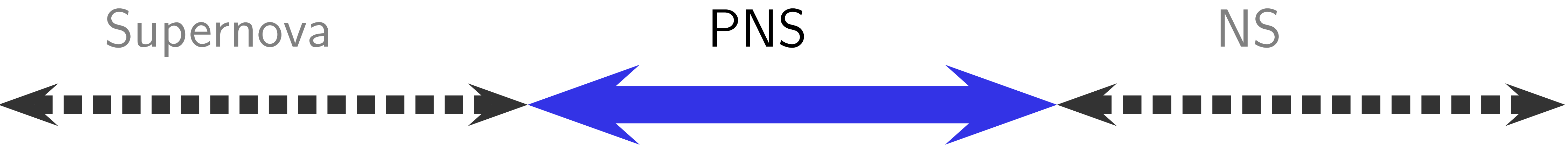
Nagakura *et al.*(2018)

- **1 model(axisymmetry)** $\sim$ 6 month by supercomputer
  $\rightarrow$ 3D simulations in Fugaku Iwakami+ '20

· Dynamical simulation ($\sim$ 10 [sec]) $R \sim 50$ [km]

· Neutrino Cooling(Diffusion) Timescale $\sim \mathcal{O}(10)$ [sec]

Evolution of Proto-Neutron Star(PNS)



- Lagrange Evolution of PNS   Burrows&Lattimer '86, Keil+ '96, Pons+ '99
  - Neutrino Transfer
    $\rightarrow$ Variation of Electron fraction
  - Energy Equation
    $\rightarrow$ Variation of Entropy
  - Equilibrium
    (Relativistic star with spherical symmetry)

- Euler "Evolution" of **Rotating Stars in GR**

  Goussard+ '97&'98, Sumiyoshi+ '99, Strobel+ '99, Villain+ '04

  - Rotation Law: Uniform or Differential Rotation $(+\alpha)$

Equilibrium of Rotating Stars in GR

Hartle '67
Butterworth&Ipser '76
Komatsu+ '89
Gourgoulhon+ '99, $\cdots$
Uryu+ '17, Camelio+ '19

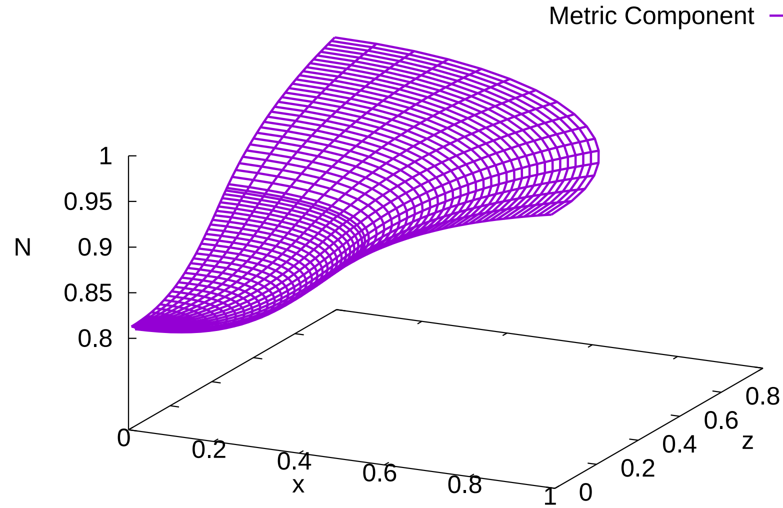
- A perfect fluid in GR (Force-Balance eq.)
  - Geometry: Second-order PDEs for $g_{\mu\nu}$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  - Fluid: First-order PDEs for $P(\text{ or } \rho)$

$$\frac{1}{\varepsilon + P}\nabla_\mu P = \nabla_\mu log(u^t) - F\nabla_\mu \Omega$$

Pressure Gradient *Gravity* *Centrifugal Force*

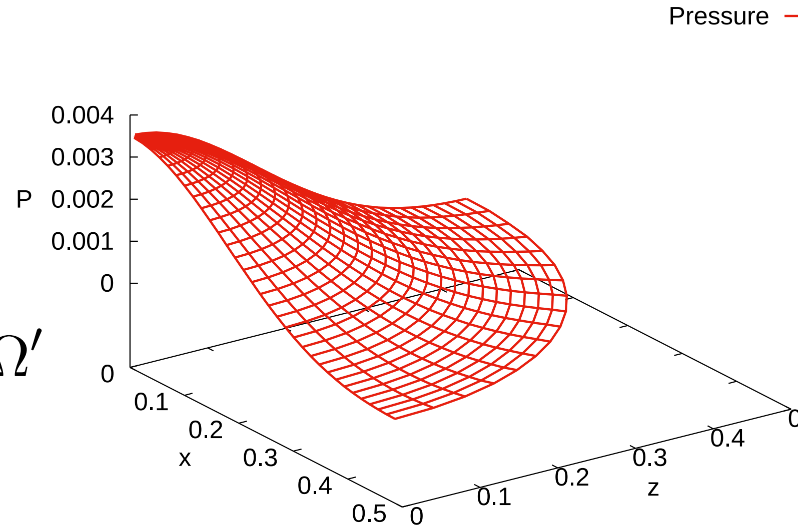


EOS: (pseudo-)Barotrope

$$H(p) \equiv \int_0^P \frac{\mathrm{d}P'}{\varepsilon(P') + P'}$$

Rotation Law

$$\mathcal{F}(\Omega) \equiv \int_{\Omega_c}^{\Omega} F(\Omega')\mathrm{d}\Omega'$$

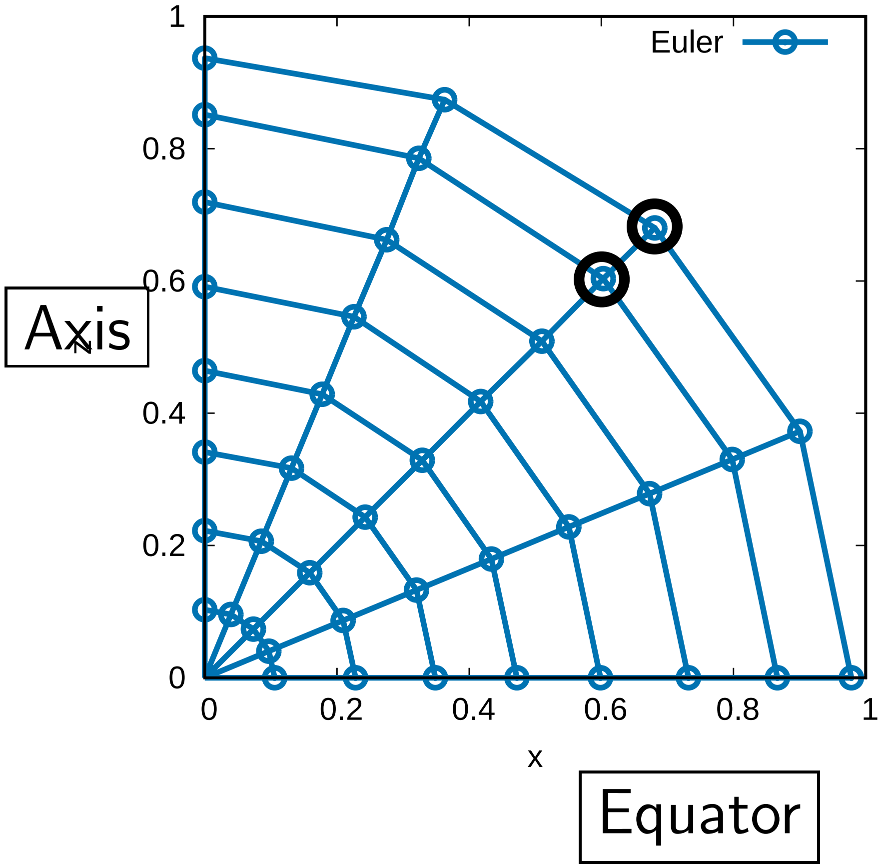


$\Rightarrow$ $$\nabla_\mu \left(H(p) - log(u^t) + \mathcal{F}(\Omega)\right) = 0.$$

One of the Einstein eqs.

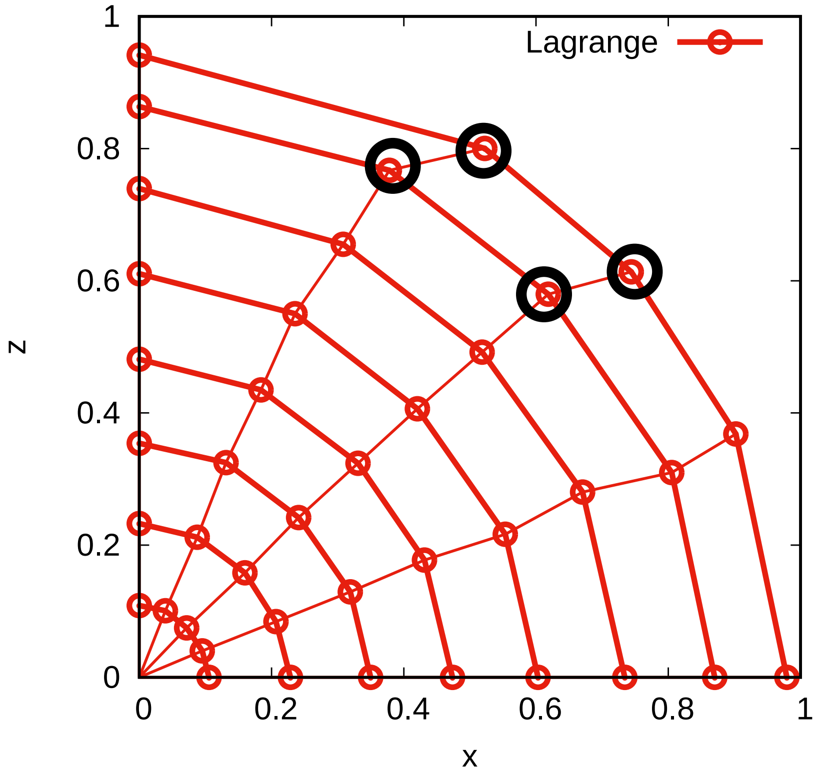
$$
\begin{aligned}
0 &\equiv G_{tt} - \frac{8\pi G}{c^4} T_{tt} \\
&= \frac{A^2\omega^2\sin^2\theta}{NB^2}\left(r^2\frac{\partial^2 N}{\partial r^2} + \frac{\partial^2 N}{\partial\theta^2}\right) + \left(\frac{N^2 A}{r^2} - \frac{A\omega^2\sin^2\theta}{B^2}\right)\left(r^2\frac{\partial^2 A}{\partial r^2} + \frac{\partial^2 A}{\partial\theta^2}\right) \\
&\quad + \frac{N^2A^2}{r^2B^2}\left(r^2\frac{\partial^2 B}{\partial r^2} + \frac{\partial^2 B}{\partial\theta^2}\right) - \frac{A^2\omega\sin^2\theta}{B^2}\left(r^2\frac{\partial^2\omega}{\partial r^2} + \frac{\partial^2\omega}{\partial\theta^2}\right) \\
&\quad + \frac{A^2\omega\sin^2\theta}{NB^2}\left(r^2\frac{\partial N}{\partial r}\frac{\partial\omega}{\partial r} + \frac{\partial N}{\partial\theta}\frac{\partial\omega}{\partial\theta}\right) + \frac{3A^2\omega\sin^2\theta}{B^3}\left(r^2\frac{\partial B}{\partial r}\frac{\partial\omega}{\partial r} + \frac{\partial B}{\partial\theta}\frac{\partial\omega}{\partial\theta}\right) \\
&\quad - \left(\frac{N^2}{r^2} - \frac{\omega^2\sin^2\theta}{B^2}\right)\left\{r^2\left(\frac{\partial A}{\partial r}\right)^2 + \left(\frac{\partial A}{\partial\theta}\right)^2\right\} - \frac{2N^2A^2}{r^2B^2}\left\{r^2\left(\frac{\partial B}{\partial r}\right)^2 + \left(\frac{\partial B}{\partial\theta}\right)^2\right\} \\
&\quad - \left(\frac{A^2\sin^2\theta}{4B^2} + \frac{3r^2\omega^2\sin^4\theta}{4N^2B^4}\right)\left\{r^2\left(\frac{\partial\omega}{\partial r}\right)^2 + \left(\frac{\partial\omega}{\partial\theta}\right)^2\right\} \\
&\quad + \frac{rA^2\omega^2\sin^2\theta}{NB^2}\frac{\partial N}{\partial r} + \left(\frac{N^2A}{r} - \frac{rA\omega^2\sin^2\theta}{B^2}\right)\frac{\partial A}{\partial r} + \frac{2N^2A^2}{rB}\frac{\partial B}{\partial r} \\
&\quad - \frac{4rA^2\omega\sin^2\theta}{B^2}\frac{\partial\omega}{\partial r} + \frac{2N^2A^2\cos\theta}{r^2B\sin\theta}\frac{\partial B}{\partial\theta} - \frac{3A^2\omega\sin\theta\cos\theta}{N^2}\frac{\partial\omega}{\partial\theta} \\
&\quad - \frac{8\pi G}{c^4}\left[(\Omega-\omega)^2\varepsilon\omega^2r^4\sin^4\theta + \left(\Omega^2P + 2\Omega\omega\varepsilon - 2\omega^2\varepsilon\right)N^2B^2r^2\sin^2\theta + \varepsilon N^4B^4\right]
\end{aligned}
$$

Euler and Lagrange formulations



$$\rho_{i,j} \equiv \frac{\Delta m_{i,j}}{\Delta V_{i,j}}$$

$\Delta m, \Delta j, \Delta s, Y_e$ attached to a fluid element



Discretization

$$\frac{\partial P}{\partial r} \sim \frac{P_{i,j} - P_{i-1,j}}{r_{i,j} - r_{i-1,j}}$$

PDEs

→Nonlinear system eqs.

Standard solver for them like Newton-Raphson method does not work. (from 1970s...)

Newton-Raphson method

To find the roots of single-variable function:

$$f(x) = 0.$$

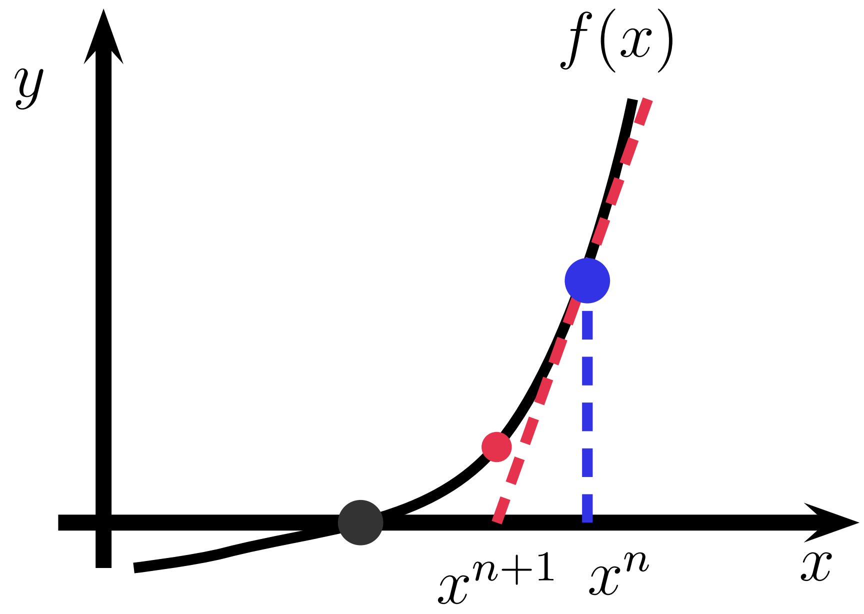
Taylor expansion
around the solution $x^* = x + \Delta x$:

$$f(x + \Delta x) = f(x) + f' \Delta x + \mathcal{O}(\Delta x^2).$$

Since $f(x + \Delta x) = 0$,we define the map:

$$\Delta x = -\frac{f(x)}{f'},$$

$$x^{n+1} \equiv x + \Delta x = x^n - \frac{f}{f'}.$$



Linearization:
$y - f(x^n) = f'(x^n)(x - x^n)$

New approximated solution is expected to be closer to the solution. A series of x^n converges to the solution.

System of nonlinear eqs.(NR and W4)

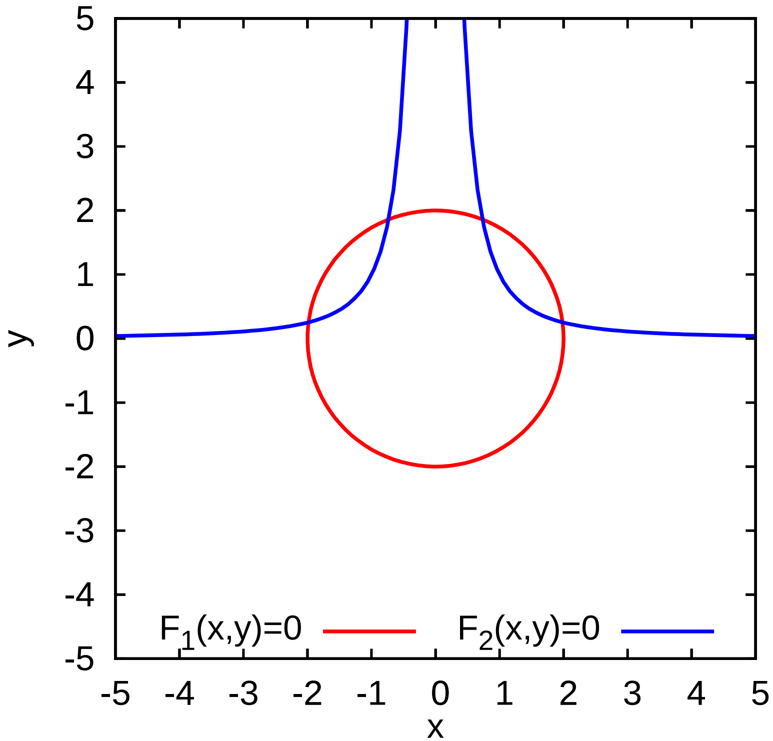
(ex) Cross-section
between circle and curves

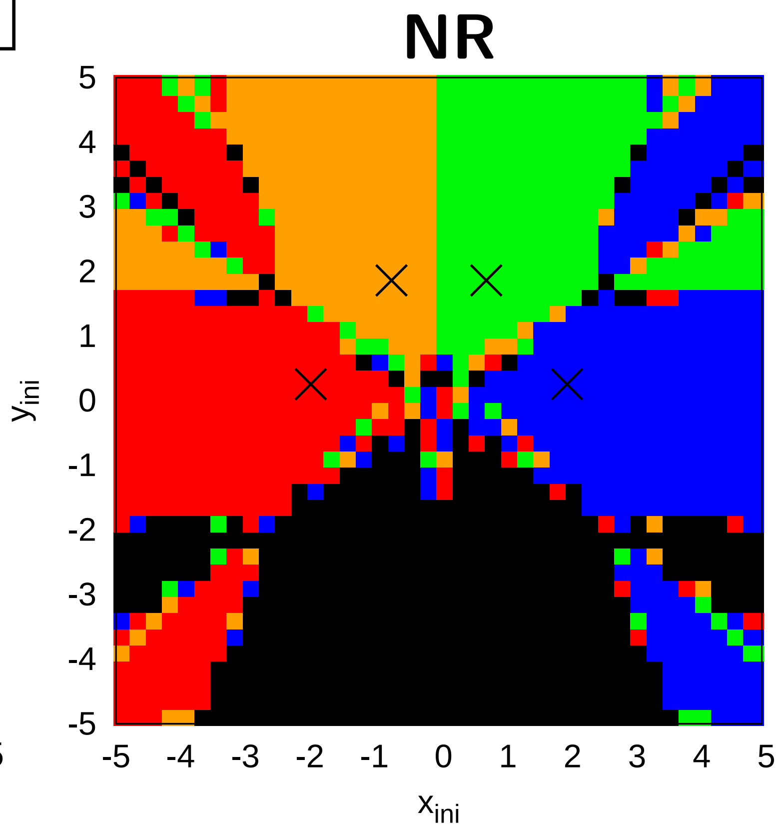
$$F_1(x,y) := x^2 + y^2 - 4 = 0,$$
$$F_2(x,y) := x^2 y - 1 = 0.$$

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

Newton-Raphson method

$$\left(\frac{\boldsymbol{x}^{n+1} - \boldsymbol{x}^n}{\Delta\tau} = -J^{-1}\boldsymbol{F} \right) \qquad \dot{\boldsymbol{x}} = -J^{-1}\boldsymbol{F}$$





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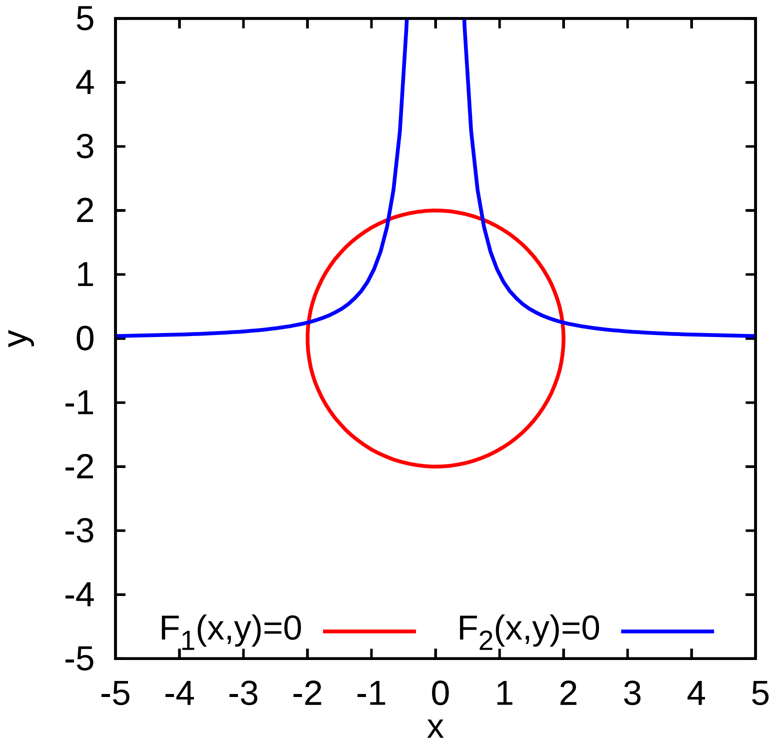


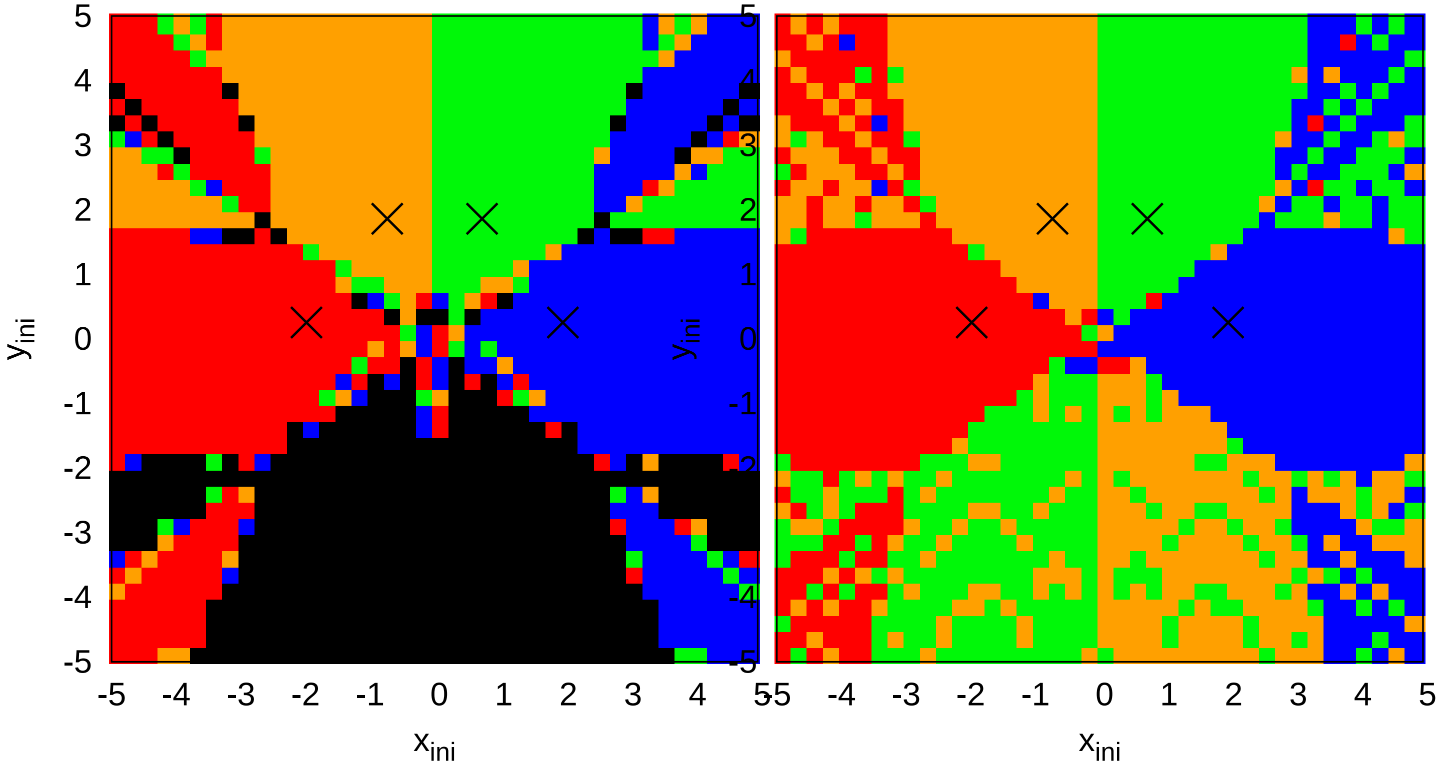
Newton-Raphson method

$$\left(\frac{\boldsymbol{x}^{n+1}-\boldsymbol{x}^n}{\Delta\tau}=-J^{-1}\boldsymbol{F}\right) \qquad \dot{\boldsymbol{x}}=-J^{-1}\boldsymbol{F}$$

W4 method (Okawa *et al.* '18) ($Y=X^{-1}J^{-1}$)

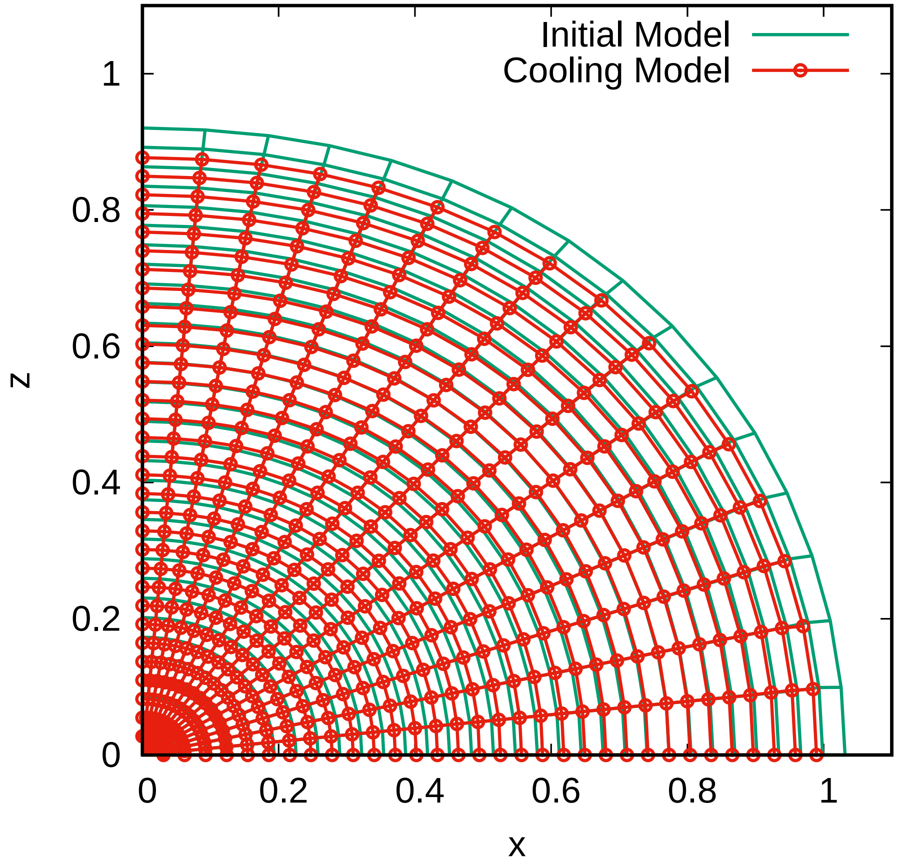
$$\dot{\boldsymbol{x}}=X\boldsymbol{p},$$
$$\dot{\boldsymbol{p}}=-2\boldsymbol{p}-Y\boldsymbol{F}.$$

NR W4



Cooling of Rotating Star

- Initial model (Green curves)
  - EOS: Polytrope
    $P = K_0 \rho^2$
  - Rotation Law: Differential
    (Constant-j law)

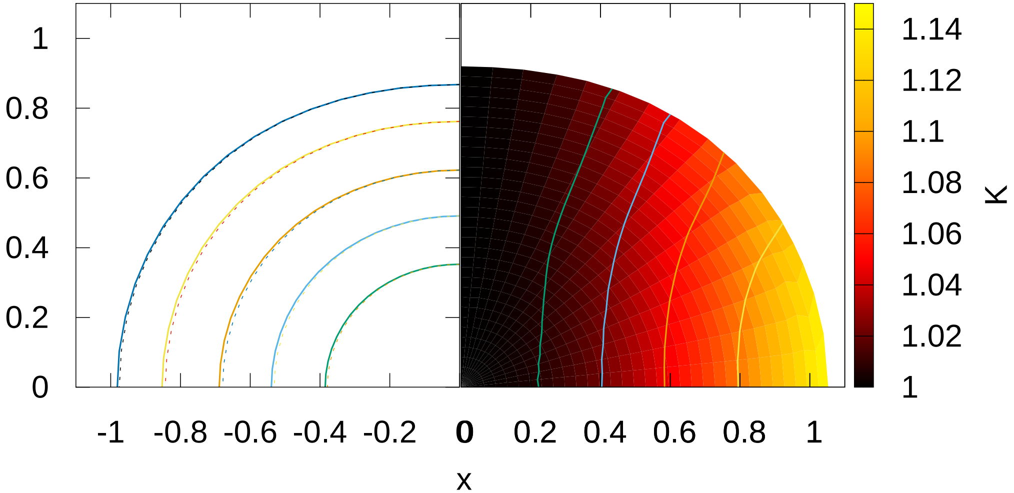


- Evolution model (Red curves)
  - Δm and Δj are Fixed.
  - Temperature variation: $P = K(t)\rho^2$
  - Specific angular momentum: $j_\varphi = h u_\varphi$ (Axisymmetry)

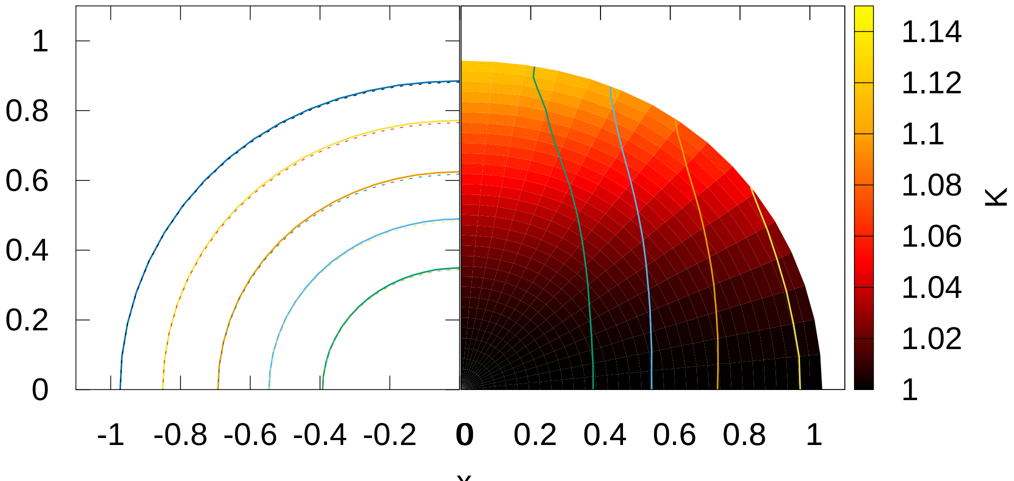
Cf. $\ell_\varphi \equiv -j_\varphi / j_t$: $\ell_\varphi = \dfrac{(\Omega - \omega) r^2 \sin^2\theta}{N^2 + \omega(\Omega - \omega) r^2 \sin^2\theta}$

Various models for Rotating Stars in GR

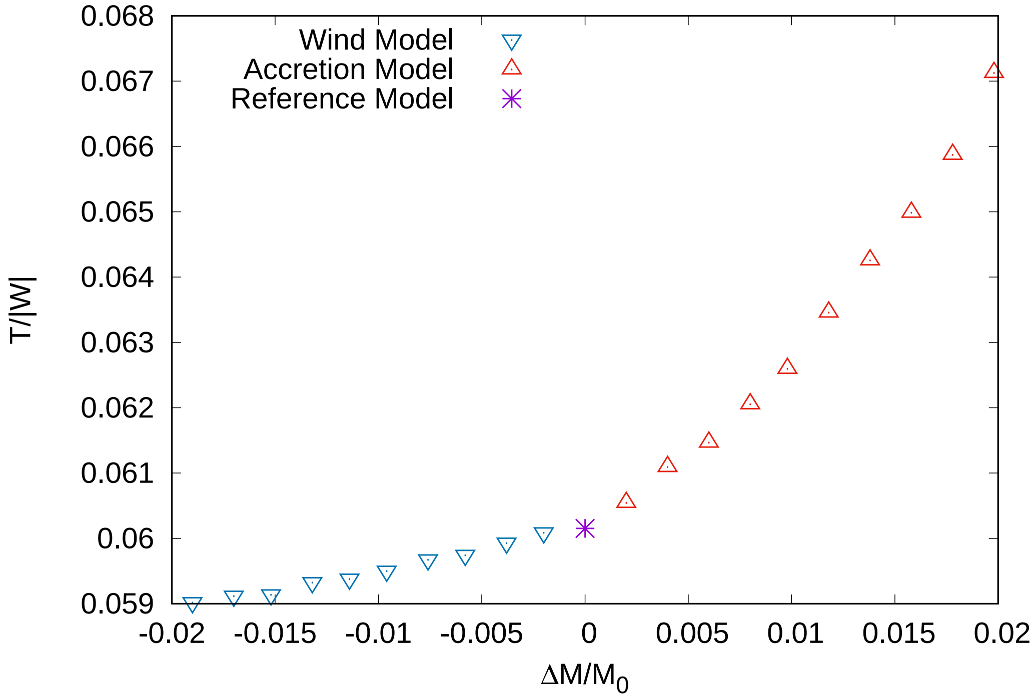
Any EOS: (ex) $P(\rho, s(r,\theta)) = K(r,\theta)\rho^2$



$$K(r,\theta) = K_0 \left(1 + \epsilon \frac{r^2}{R_e^2} \sin^2\theta\right)$$



Accretion or Mass-loss



Lagrange Evolution!!

Summary

- Equilibrium of rotating stars in GR
  = Solving the system of nonlinear eqs.
  - PDEs: system of nonlinear eqs.
  - Newton-Raphson method needs fine-tuned initial guess.
- New method (W4) allows us to construct it!
  - Lagrange formulation for the evolution
  - Let us solve a realistic system!
    (Neutrino transport, gravitational waves, EOS?)