Kerr-Newman-de Sitter QNMs and Strong Cosmic Censorship Conjecture

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Summary

K(N)dS Metric

- The KN-dS metric in Boyer-Lindquist coordinates is given by (using $G = c = \hbar = 1$)

$$ds^2 = \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) - \frac{\Delta_r}{\Xi^2 \rho^2} \left(dt - a \sin^2 \theta d\phi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} \left[a dt - (r^2 + a^2) d\phi \right]^2 , \quad (1)$$

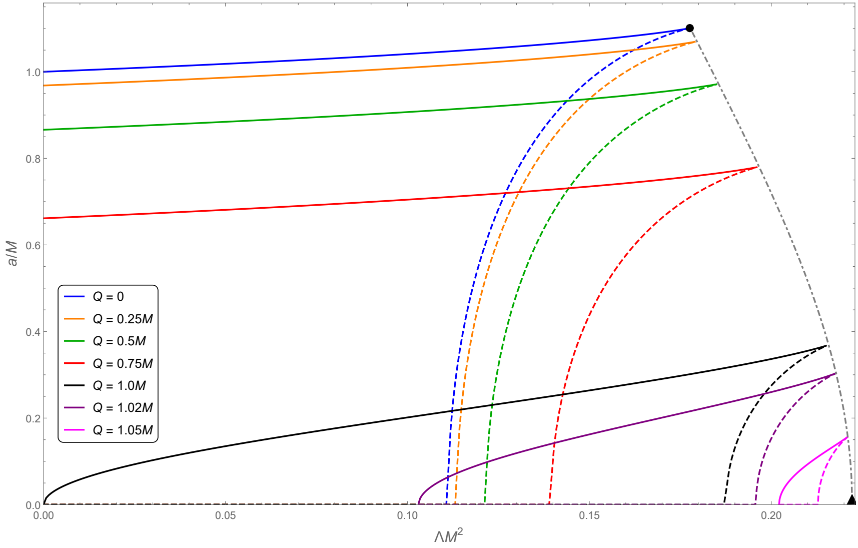
where $\Lambda = 3/L^2$ and

$$\Delta_r = (r^2 + a^2) \left(1 - \frac{r^2}{L^2} \right) - 2Mr + Q^2 = -\frac{r - r_{\rm n}}{L^2} (r - r_-)(r - r_+)(r - r_c), \quad (2)$$

$$\Delta_\theta \equiv 1 + \alpha \cos^2 \theta, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \alpha \equiv \Xi - 1 \equiv a^2/L^2. \quad (3)$$

- The roots r_-, r_+ and r_c are respectively the Cauchy, event and Cosmological horizons. $r_{\rm n}$ is an inner Cosmological horizon and $r_{\rm n} = -(r_+ + r_- + r_c)$.

K(N)dS Black Holes



Master Equations for Field Perturbations in K(N)dS

- A massless conformally-coupled scalar field Ψ_0 with charge q obeys the following equation [Konoplya and Zhidenko, 2007]:

$$(\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu)\Psi_0 = \frac{4\Lambda}{6}\Psi_0, \tag{4}$$

where ∇ and A_μ are, respectively, the covariant derivative and the electromagnetic potential, this given by

$$\boldsymbol{A} = \frac{-Qr}{\Xi\rho^2}(dt - a\sin^2\theta d\varphi). \tag{5}$$

- In its turn, a massless Dirac spinor field $\Psi_{1/2}$ with charge q obeys the equation [Liu, Van Vooren, Zhang, and Zhong, 2019]:

$$\gamma^\mu(\partial_\mu + \Gamma_\mu - iqA_\mu)\Psi_{1/2} = 0, \tag{6}$$

where γ^μ and Γ_μ are the Dirac gamma matrices and the spin connection matrices, respectively.

- Both equations for charged scalar and Dirac fields can be separated by variables using a mode decomposition as

$$\Psi_{|s|}(\mathrm{x}) = \int_{\mathbb{R}} d\omega \sum_{\ell=|s|}^{\infty} \sum_{m=-\ell}^{\ell} {}_{|s|}c_{\ell m\omega}\, e^{-i(\omega t - m\varphi)}\, {}_{|s|}\psi_{\ell m\omega}(r,\theta) \tag{7}$$

  for $s = 0,\ \pm 1/2$, where ${}_{|s|}c_{\ell m\omega} \in \mathbb{C}$ are constant coefficients, ω is the mode frequency, m is the azimuthal number and ℓ is the multipolar number.
- We can use ${}_0\psi_{\ell m\omega} = R_0(r)S_0(\theta)$ for spin 0 and the 4-spinor ${}_{1/2}\psi_{\ell m\omega}$ can have its components separated as $f_\pm(r,\theta)R_{\pm 1/2}(r)S_{\pm\varsigma 1/2}(\theta)$ where $f_\pm(r,\theta) = \Delta_r^{\pm 1/4}(r - i\varsigma\cos\theta)^{-1/2}$ [$\varsigma = 1(-1)$ for left(right)-handed spinors].
- The radial R_s and angular S_s factors satisfy master ordinary differential equations (ODEs), in the sense that the spin $s = 0,\ \pm 1/2$ appears as a parameter.

Master Spin-s Equations

- Namely, the master radial ODE is

$$\left[\Delta_r^{-s}\partial_r\Delta_r^{s+1}\partial_r + \frac{W^2 - isW\Delta_r'}{\Delta_r} + 2isW' - Y\right]R_s(r) = 0, \tag{8}$$

- and the master angular ODE is

$$\left[\partial_u\Delta_u\partial_u - \frac{1}{\Delta_u}\left(H + \frac{s}{2}\Delta_u'\right)^2 + 2sH' - X\right]S_s(u) = 0, \tag{9}$$

with $W(r) \equiv \Xi[\omega(r^2+a^2) - am] - qQr$, $Y \equiv \frac{2}{L^2}(s+1)(2s+1)r^2 + {}_s\lambda_{\ell m}$, $H(u) \equiv \Xi[a\omega(1-u^2) - m]$, $\Delta_u(u) \equiv (1-u^2)(1+\alpha u^2)$ and $X \equiv 2(2s^2+1)\alpha u^2 - {}_s\lambda_{\ell m} - s(1-\alpha)$.

- We have introduced a new coordinate $u \equiv \cos\theta$ and ${}_s\lambda_{\ell m} = {}_s\lambda_{\ell m}(\omega)$ is the separation constant that decouples $R_s(r)$ and $S_s(\cos\theta)$.
- For the KdS case ($Q = 0$), Eqs. (8) and (9) are also the pair of ODEs for the spin $s = \pm 1$ (electromagnetic) and $s = \pm 2$ (gravitational) perturbations, as can be seen in [Suzuki, Takasugi, and Umetsu, 1998].

The Angular Eigenvalues

- Our main objective is to obtain angular and radial eigenvalues of the ODEs Eqs. (8) and (9), associated with the following boundary conditions:
- We choose boundary conditions such that the solutions of the angular ODE (9) are regular at the two endpoints $u = \pm 1$.
- Then the separation constant ${}_s\lambda_{\ell m} = {}_s\lambda_{\ell m}(\omega)$ becomes our wanted angular eigenvalue.
- Mathematically, we ask

$$S_s(u) \to \begin{cases} (u+1)^{|m-s|/2} & \text{as } u \to -1, \\ (u-1)^{|m+s|/2} & \text{as } u \to 1. \end{cases} \tag{10}$$

Mode Frequencies

- We choose boundary conditions such that the solutions of the radial ODE (9) are purely ingoing at the (future) event horizon ($r = r_+$) and outgoing boundary condition at the (future) Cosmological horizon ($r = r_+$).
- From that, we obtain the mode frequencies $\omega = \omega_{\ell m n}$.
- Mathematically, we ask

$$R_s(r) \to \begin{cases} (r - r_+)^{-s} e^{-i(\omega_{\ell m n} - m\Omega_+ - q\phi_+) r_*}, & \text{as } r \to r_+, \\ e^{+i(\omega_{\ell m n} - m\Omega_c - q\phi_c) r_*}, & \text{as } r \to r_c, \end{cases} \tag{11}$$

where $dr_* \equiv \Xi(r^2 + a^2) dr/\Delta_r$, and we also define

$$\Omega_j \equiv \frac{a}{r_j^2 + a^2}, \quad \phi_j \equiv \frac{Q r_j}{\Xi(r_j^2 + a^2)}, \quad \kappa_j \equiv \frac{|\Delta_r'(r_j)|}{2\,\Xi\,(r_j^2 + a^2)}, \; j = \{\mathrm{n}, -, +, c\}, \tag{12}$$

as the angular velocity, electric potential and surface gravity, respectively, at the horizon $r = r_j$.

Numerical Method

- In order to numerically obtain the angular eigenvalues and mode frequencies, we extended to KNdS the method that Leaver's [Leaver, 1985] originally developed and applied to Kerr spacetime and was later extended to Kerr-dS in [Yoshida, Uchikata, and Futamase, 2010].
- The method consists of carrying out a power series expansion of the angular solution $S_s(u)$ essentially about $u = -1$ (or $u = 1$) and then requiring regularity at $u = 1$ (or $u = -1$); this leads to a continued fraction equation for the eigenvalue ${}_s\lambda_{\ell m}(\omega)$.
- Similarly, expanding the radial solution $R_s(r)$ essentially about the cosmological horizon $r = r_c$ obeying the boundary condition in Eq. (11) as $r \to r_c$, and then requiring the boundary condition as $r \to r_+$ in Eq. (11) leads to a continued fraction equation (which involves ${}_s\lambda_{\ell m}$) for the mode frequency $\omega_{\ell m n}$.
- We will not show the development of the these continued fractions here.

Solving the Angular and Radial Continued Fractions

- In order to find both ${}_s\lambda_{\ell m}$ and $\omega_{\ell mn}$ efficiently, we need some initial guess to delimit the region we want to obtain our mode frequencies.
- [Suzuki, Takasugi, and Umetsu, 1998] provided an expansion for ${}_s\lambda_{\ell m}$ for $|a\omega| \ll 1$ and $|a|/L \ll 1$. Partitioning into small sizes of $|a|$, we can obtain ${}_s\lambda_{\ell m}(a\omega)$ for any $a\omega$.
- As for $\omega_{\ell mn}$, we can use analytical results for QNMs as our initial guesses.
- We can also take advantage of the fact that the radial continued fraction is an analytical function of ω and look for its zeros in a complex-frequency plane. This is a slow method and we do it only when no analytical approximation is good.

Summary

Accessory Parameter Expansion: The Main Idea

In the Accessory Parameter Expansion technique, the accessory parameter of a Heun equation [Novaes and Carneiro da Cunha, 2014, Novaes, Marinho, Lencsés, and Casals, 2019] is written in two ways:

- In terms of the monodromy data (which has information about the near-singular-point behavior of the ODE as well as the BCs for this ODE; in this case, the accessory parameter is expressed in terms of a isomonodromic τ-function [Lencsés and Novaes, 2018]);
- And in terms of the original radial/angular ODEs written in a Heun-like equation.

We have presented in great detail in [Novaes, Marinho, Lencsés, and Casals, 2019] how to obtain both angular eigenvalues and QNMs via the accessory parameter expansion (APE) of Kerr-de Sitter spacetime near the Nariai limit ($r_+ \to r_c$). Here we just present the results and numerical comparison for the QNMs.

APE in KdS

Using the APE, we were able to find that QNMs near $m\Omega_+$ and $m\Omega_c$, behave as

$$\omega_{\mathrm{NL},+} \equiv m\Omega_+ + \epsilon\bar{\omega}_0 + \epsilon^2\bar{\omega}_1 \text{ and } \omega_{\mathrm{NL},c} \equiv m\Omega_c - \epsilon(\bar{\omega}_0)^* + \epsilon^2(\bar{\omega}_1)^*, \tag{13}$$

where $\epsilon = (r_c - r_+)/L$ and (with $\bar{\kappa} \equiv \lim_{\epsilon\to 0} \frac{\kappa_+ + \kappa_c}{\epsilon}$, $\tilde{\Omega} \equiv \frac{1}{\bar{\kappa}} \lim_{\epsilon\to 0} \frac{\Omega_+ - \Omega_c}{\epsilon}$)

$$\bar{\omega}_0 = \frac{\bar{\kappa}}{2}\left[-i\left(n+\frac{1}{2}\right) - m\tilde{\Omega} + \sqrt{\bar{\lambda}_0 + m^2\tilde{\Omega}^2 - \left(s+\frac{1}{2}\right)^2}\right]. \tag{14}$$

(with $\bar{\lambda}_0 \equiv \frac{2(s+1)(2s+1)\bar{r}_+^2 + {}_s\lambda_{\ell m}(am/[a^2+\bar{r}_+^2])L^2}{(\bar{r}_+ - \bar{r}_-)(3\bar{r}_+ + \bar{r}_-)}$) and (with $\tilde{\omega}_0 \equiv 2\bar{\omega}_0/\bar{\kappa} + m\tilde{\Omega}$):

$$\begin{aligned}\bar{\omega}_1 = \frac{\left[4\Xi(a^2+\bar{r}_+^2)\right]^{-1}}{\tilde{\omega}_0 + i(n+1/2)} &\left[\frac{L\bar{\kappa}}{2}\tilde{\omega}_0 \frac{d}{d\omega}\, {}_s\lambda_{\ell m}\left(\frac{am}{a^2+\bar{r}_+^2}\right) + \right.\\ &\left. + im\tilde{\Omega}(2n+1)\bar{r}_+\left(2 - L\Xi\bar{\kappa}\right) + \frac{m\tilde{\Omega}\left(\bar{r}_+ + \bar{r}_-\right)^2\left(2s^2+s-\bar{\lambda}_0\right)\tilde{\omega}_0}{2\bar{r}_+(\bar{\lambda}_0 + m^2\tilde{\Omega}^2 - s^2 - s)}\right],\end{aligned} \tag{15}$$

Numerical Calculation of Near Nariai Limit QNMs I

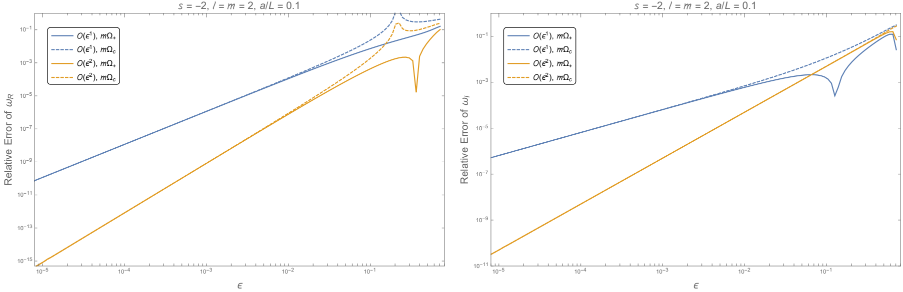


Figure: Relative error in the QNM frequencies when comparing our analytic expansion in Eq. (13) with a numerical calculation. It is plotted as function of ϵ and for the mode $s = -2$, $\ell = m = 2$ and $a = 0.1L$. The left graph is for the real part of the QNM frequency and the right graph for the imaginary part.

Numerical Calculation of Near Nariai Limit QNMs II

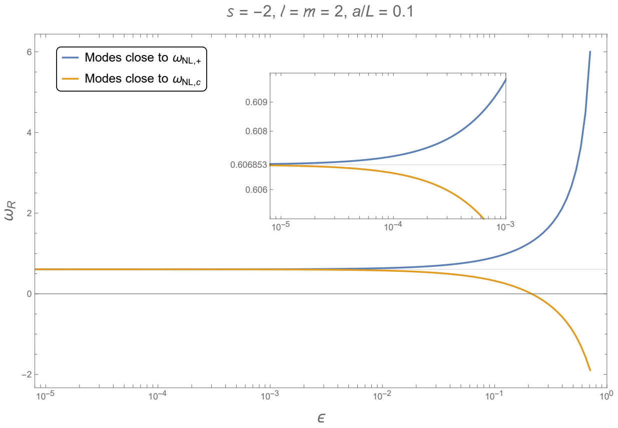


Figure: Real part of the numeric QNMs as we approach the Nariai limit: the real parts of the modes that are close to $\omega_{\mathrm{NL},+}$ and the ones that are close to $\omega_{\mathrm{NL},c}$ both tend to $m\Omega_+ = m\Omega_c \approx 0.606863$ as $\epsilon \to 0$. Not shown: imaginary part going to zero as $\epsilon \to 0$.

Summary

Strong Cosmic Censorship Conjecture

- The strong cosmic censorship (SCC) conjecture asserts that, for generic asymptotically flat initial data for Einstein's equationCauchy horizons do not form [Penrose, 1979].
- The modern statement of the SCC is that, although it may be possible to extend the metric continuously across the Cauchy horizon, generically it should not be possible to do so with locally square integrable Christoffel symbols. [Christodoulou, 2008]
- Some time ago, it was observed that the mechanism behind this instability is weaker when the cosmological constant Λ is positive. It was shown [Cardoso *et al.*, 2018a] that in Reissner-Nordström-de Sitter black hole SCC could be violated when considered a massless uncharged scalar field.
- This was shown via a quantity that controls the stability of the Cauchy horizon (at linear level), named β, defined via

$$\beta \equiv \inf\left\{-\Im(\omega_{s\ell m})\right\}/\kappa_{-}; \tag{16}$$

where $\inf\left\{-\Im(\omega_{s\ell m})\right\}$ the spectral gap: the most slowly decaying quasinormal mode frequency of the black hole.

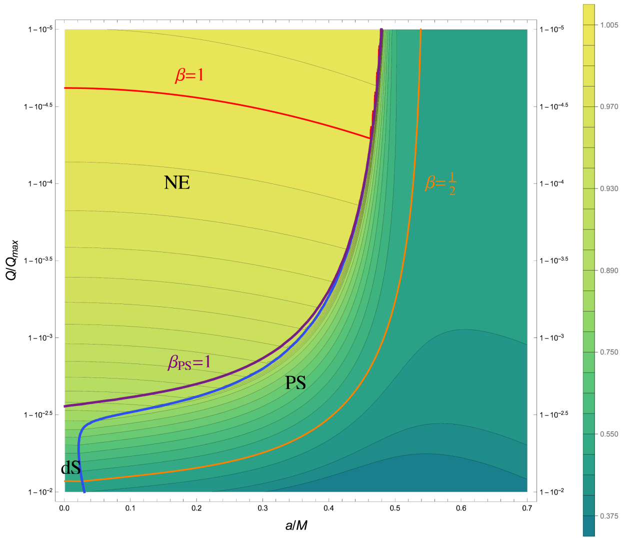
Strong Cosmic Censorship Conjecture

- Lately, a lot has been studied with respect to the SCC violation, and it so far what we have are:
  - At linear level, SCC is violated in massless uncharged scalar [Cardoso *et al.*, 2018a] and Dirac [Ge *et al.*, 2019] field in RNdS.
  - At linear level, SCC is saved if the field has sufficient large charge or mass [Cardoso *et al.*, 2018b] in the case of RNdS.
  - KdS black holes do not violate SCC [Dias *et al.*, 2018].
  - KNdS black holes with a sufficient high rotation parameter do not violate SCC [Hod, 2018].
- The non-linear evolution of a massive/charged scalar field, however, SCC another ingoing discussion [Luna *et al.*, 2021]. But, so far, [Zhang and Zhong, 2019] seems to find that the SCC is preserved when considering the cases the linear level states violation of SCC.
- Because near an extremal RNdS black hole the SCC is violated, but in Kerr-de Sitter it isn't, there should be at least a region with small rotation where the SCC could be violated in KN-dS.

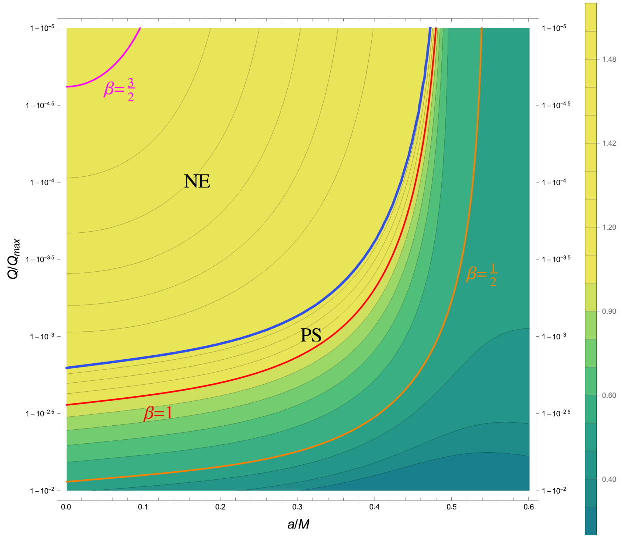
Strong Cosmic Censorship Conjecture

- In the main text, we have discussed how to obtain the SCC conditions in terms of β for both spin-0 and and spin-1/2 charged fields in KNdS.
- Our main result is that $\beta > 1/2$ corresponds to local integrability of the stress-energy tensor at the Cauchy horizon in both cases (spin 0 and 1/2).
- Also the stress-energy tensor is C^1 (at the Cauchy horizon, i.e. the condition to boundedness) for $\beta > 1$ in the scalar field case, and $\beta > 3/2$ in the Dirac field case.
- We now show the cases where we have found glimpses of SCC violation in KNdS space-time considering both uncharged and charged scalar/Dirac field perturbations.

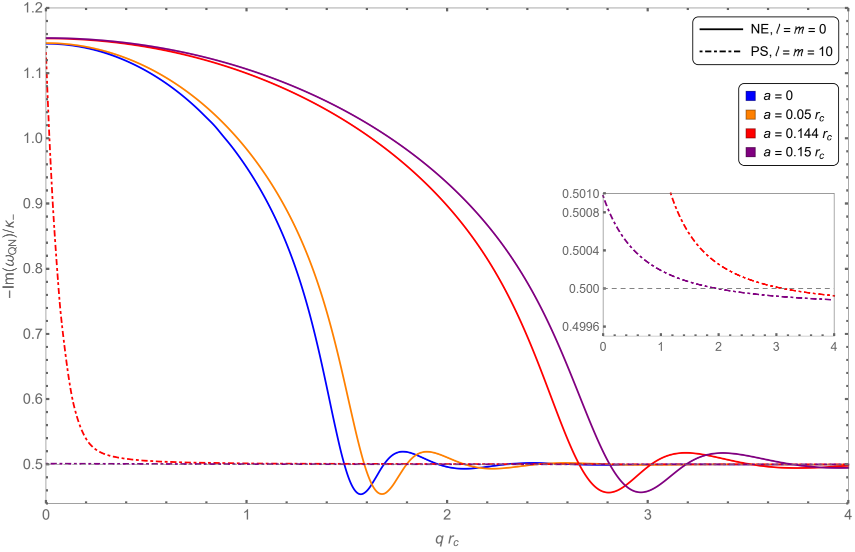
$s = 0, q = 0, \Lambda M^2 = 0.02$



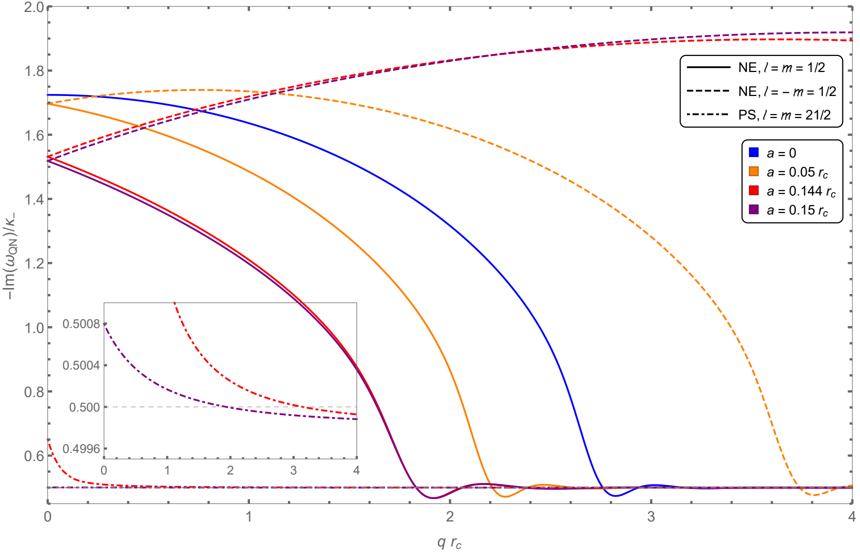
$s = 1/2, q = 0, \Lambda M^2 = 0.02$



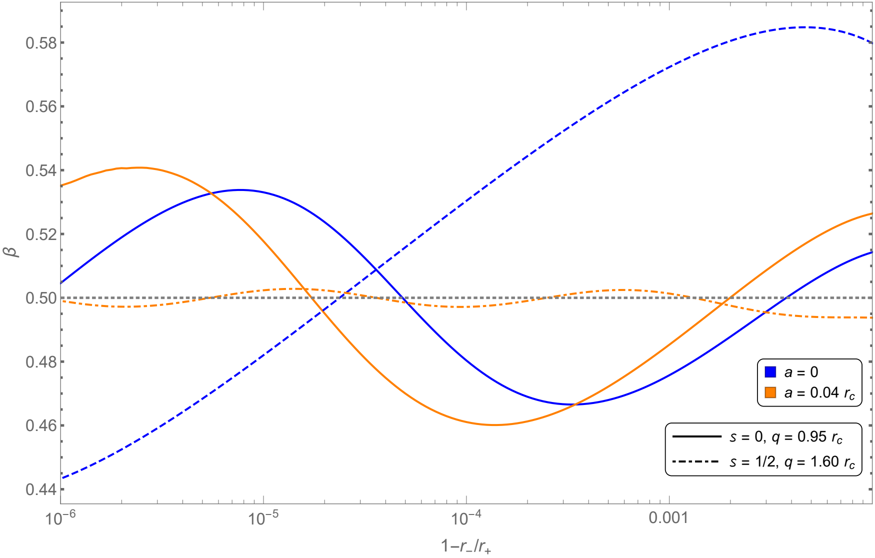
$s = 0, r_+ = r_c/3, Q = (1 - 10^{-4})Q_{\text{max}}$



$s = 1/2, r_+ = r_c/3, Q = (1 - 10^{-4})Q_{\text{max}}$



$r_+ = r_c/2, s = 0$ and $s = 1/2$



Summary

Remarks

- Our main objective in this Ph.D. is to make a comprehensive search of mode frequencies (QNMs and unstable modes) in KNdS and KdS space-times and to study the fate of SCC in KNdS when considering different field perturbations.
- Here we showed part of this goal showing the behaviour of QNMs near the extremes of K(N)dS black holes: Nariai limit (NL) and near extremal BH limit (NE).
- About QNMs in the near NL ($r_+ \to r_c$), we obtained an analytical expression for the QNMs and we also could see that in this limit we have an accumulation point of QNMs around $\omega = m\Omega_{+,c}$.
- Our study of QNMs in NE ($r_+ \to r_-$) was focused on modes where we could possibly see violation of SCC.
- For (un-)charged fields, the cases where $a < a_c^{\mathrm{KNdS}}(q)$, even for large charge we could see glimpses of violation of linear version of SCC due to the wiggles-like-behavior of QNMs in the large-q limit.

What To Do Next?

- Make a spectroscopy of QNMs in KNdS for different regions of the space-time [In progress];
- Extending the APE analysis done to KdS to KNdS [In progress].
- Make a wider look for unstable modes [To do];

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Muito obrigado a todos!

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