Singularity theorems in semiclassical gravity

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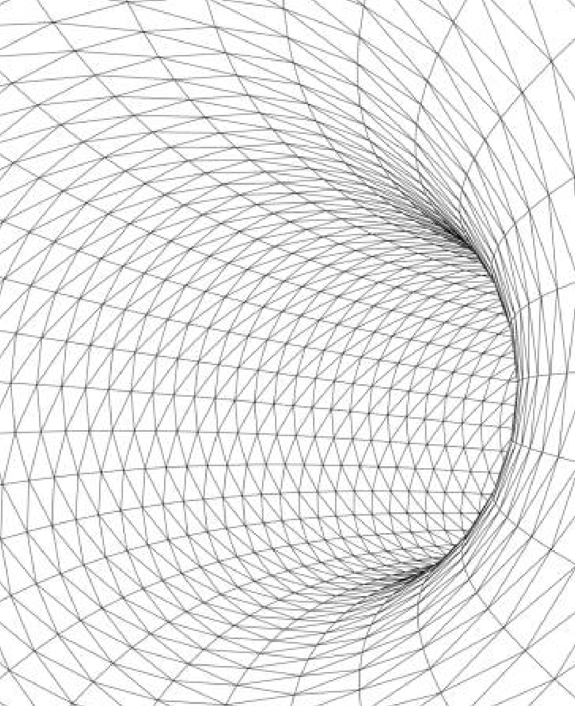

Outline

Singularity theorems structure

Definition

A spacetime is singular if it possesses at least one incomplete geodesic.

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   There exists a trapped surface (null geodesics): $H < 0$ or a spatial slice with negative expansion (timelike goedesics): $K < 0$

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2. **The energy condition**
   Restriction on the stress-energy tensor expressing "physical" properties of matter.

Null geodesics: Null energy condition (NEC) ℓ^μ: null vector
Timelike geodesics: Strong energy condition (SEC) U^μ: timelike vector

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Physical form	Geometric form	Perfect fluid
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$	$R_{\mu\nu}\ell^\mu\ell^\nu \geq 0$	$\rho + P \geq 0$
$(T_{\mu\nu} - \frac{Tg_{\mu\nu}}{n-2})U^\mu U^\nu \geq 0$	$R_{\mu\nu}U^\mu U^\nu \geq 0$	$\rho + P \geq 0$ and $(n-3)\rho + (n-1)P \geq 0$

Singularity theorems structure

3. **Causality condition**
   There is a Cauchy surface: spacelike hypersurface which intersects causal geodesics once and only once

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Proof structure

1. Initial condition: Geodesics start focusing
2. Energy condition: Focusing continues
3. Causality condition: No focal points

$\Rightarrow$ Geodesic incompleteness

From classical to semiclassical singularity theorems

Problem

Pointwise energy conditions are violated by many classical and all quantum fields

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Can we have singularity theorems with weaker energy conditions?

$$\int_\gamma f^2 R_{\mu\nu} U^\mu U^\nu \geq -(Bound)$$

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Can we have semiclassical singularity theorems?

Singularity theorems with weakened energy conditions

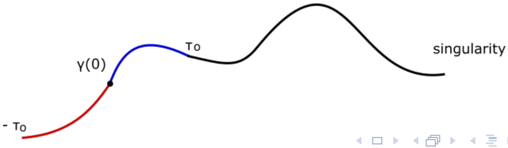
Theorem [Fewster, E-AK, 2019]

1. Energy condition

$$\int_0^\tau f(t)^2 \overbrace{R_{\mu\nu}U^\mu U^\nu}^{\rho}\, dt \geq -Q_m \|f^{(m)}\|^2 - Q_0\|f\|^2, \quad \|f\|^2 = \int_\gamma f^2 dt$$

   and Scenario 1: $\rho \geq 0$ for $[0, \tau_0]$: SEC obeyed for a short time
   or Scenario 2: $\rho < 0$ for $[-\tau_0, 0]$: SEC violated before we measure K
2. Initial condition: $K \leq -\nu(Q_m, Q_0, \tau_0, \tau)$
3. Causality condition: There exists a Cauchy surface.

$\Rightarrow$ The spacetime is timelike geodesically incomplete.



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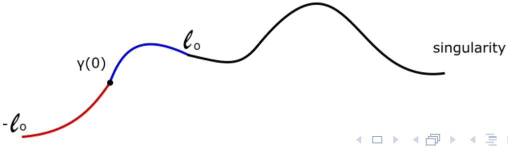
$\Rightarrow$ The spacetime is null geodesically incomplete.



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Choice of affine parameter: Choose an affine coordinate λ on γ, so that $\hat{H}_\mu d\gamma^\mu / d\lambda = 1$.

Towards semiclassical singularity theorems

(A) Prove quantum energy inequalities (QEIs) for the relevant quantities for timelike and null geodesics

Example of a QEI (bound on energy density in Minkowski spacetime)

$$\int dt\, f^2 \langle : T_{\mu\nu} U^\mu U^\nu : \rangle_\omega \geq -\frac{1}{16\pi^2} \int f''(t)^2 dt$$

[Ford, Roman, 1995], [Fewster, Eveson, 1998]

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$$8\pi G_N \langle T_{\mu\nu} \rangle_\omega = G_{\mu\nu} \, .$$

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(C) Estimate the required initial contraction for physical spacetimes

Timelike semiclassical singularity theorem

(A) Quantum strong energy inequality (QSEI) for the minimally coupled scalar field: The main observable of interest is the effective energy density (EED) (quantity appearing in the SEC)

$$\rho_U = T_{\mu\nu}\left(U^\mu U^\nu - \frac{g^{\mu\nu}}{n-2}\right)$$

$$\int_\gamma \langle :\rho_U: \rangle_\omega f^2 dt \geq -Q_m |||f|||^2 - Q_0 ||f||^2$$

$$|||f|||^2 \equiv \sum_j^m c_j(T_0,\tau)||f^{(j)}||^2 , \quad Q_m = \frac{\hbar S_{2m-2}}{(2\pi)^{2m-2}} , \quad \text{and} \quad Q_0 = \frac{4\pi M^2 \phi^2_{\max}}{m-1}$$

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- T_0: Curvature scale
- τ: Maximum time for singularity
- M: Mass of the field
- m: $n/2$
- $\phi^2_{\max}$: Maximum field value

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(B) Semiclassical Einstein equation

$$8\pi G_N \langle :\rho_U: \rangle = R_{\mu\nu} U^\mu U^\nu$$

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[Fewster, E-AK, 2021]

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$$\int dt f^2(t) R_{\mu\nu}U^\mu U^\nu \geq -Q_m|||f|||^2 - Q_0||f||^2$$

   and $R_{\mu\nu}U^\mu U^\nu \geq 0$ holds for $t \in [0, \tau_0]$

2. The initial extrinsic curvature of S satisfies

$$K \leq -\nu(M, \phi_{\max}, T_0, \tau_0, \tau)$$

3. There exists a Cauchy surface.

$\Rightarrow$ The spacetime is timelike geodesically incomplete.

Timelike semiclassical singularity theorem

(C) Estimate the required initial contraction for cosmological spacetimes

$$K \leq -\nu(M, \phi_{\max}, T_0, \tau_0, \tau)$$

Particle	ν in s^{-1}
Pion	3.57×10^{-20}
Proton	1.73×10^{-18}
Higgs	3.09×10^{-14}

- The SEC was last satisfied when $t_* = 2.41 \times 10^{17}$s,
  $H_* = 3.14 \times 10^{-18}\ \mathrm{s}^{-1}$

Null quantum energy inequalities

Can we have similar QEIs over a null geodesic?

$$\int d\lambda \langle :T_{\mu\nu}: \ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -A \int d\lambda f''(\lambda)^2$$

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The counterexample

Considered a sequence of vacuum-plus-two-particle states in which the three-momenta of excited modes are unbounded and become more and more parallel to the spatial part of the null vector ℓ^μ. [Fewster, Roman, 2002]

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Idea

In quantum field theory there is often an ultraviolet cutoff ℓ_{UV} which restricts the three-momenta. We can write $G_N \lessapprox \ell_{\mathrm{UV}}^2/N$.

Null semiclassical singularity theorem

(A) Smeared null energy condition (SNEC) for the minimally coupled scalar field in four dimensional Minkowski spacetime

$$\int d\lambda \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega f^2(\lambda) \geq -Q_1 \|f'\|^2$$

$$Q_1 = \frac{4B}{G_N}$$

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What is B?

In order to saturate SNEC, we need to saturate the inequality $NG_N \lessapprox \ell_{\mathrm{UV}}^2$ which gives $B = 1/32\pi$. Not saturated in controlled constructions: the UV cutoff of the theory is far from Planck scale so it is well-motivated to consider $B \ll 1$.

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(B) Semiclassical Einstein equation

$$8\pi G_N \langle :T_{\mu\nu}:\ell^\mu \ell^\nu \rangle_\omega = R_{\mu\nu}\ell^\mu \ell^\nu$$

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[Freivogel, E-AK, Krommydas, 2020]

1. Energy condition

$$\int d\lambda f^2(\lambda) R_{\mu\nu}\ell^\mu \ell^\nu \geq -Q_1 ||f'||^2$$

   and $R_{\mu\nu}\ell^\mu\ell^\nu \leq 0$ holds for $\lambda \in [-\ell_0, 0]$
2. The mean normal curvature of P satisfies

$$H \leq -\nu(B, \ell_0, \ell)$$

3. There exists a non-compact Cauchy surface.

$\Rightarrow$ The spacetime is null geodesically incomplete.

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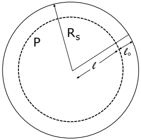
(C) Estimate the required mean normal curvature for evaporating black holes



Affine distance → Coordinate distance

- $\ell \to yR_s$
- $\ell_0 \to xR_s$

Strategy: compare H of Schwarzschild geometry to $\nu(B, \ell_0, \ell)$ from theorem

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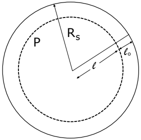

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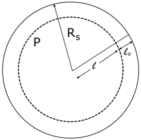

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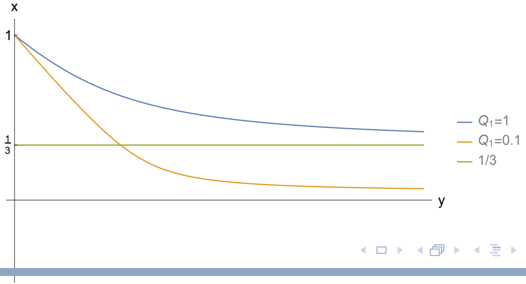

Motivation

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- Satisfies the Fewster-Roman counterexample ✔
- Proof for free fields in Minkowski [Fliss, Freivogel, 2021] ✔
- Curved spacetimes, interacting fields ✗
- Limit of $\ell_{\mathrm{UV}} \to 0$ ✗

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Idea

Since the bound when we smear in one null direction is divergent, smear in two null directions $x^{\pm}$

The double smeared null energy condition (DSNEC)

[Fliss, Freivogel, E-AK, 2021]
For free scalar fields in 4-dimensional Minkowski

$$\int d^2x^{\pm} f(x^+,x^-)^2 \langle T_{--}\rangle_\omega \geq -\frac{16}{81\pi^2}\left(\int dx^+ (f_+''(x^+))^2\right)^{1/4} \times \left(\int dx^- (f_-''(x^-))^2\right)^{3/4}.$$

Advantages

- Rigorously proven from a general QEI
- Can be generalized to curved spacetimes
- The smearing can be controlled and does not depend on the theory

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Question

Can we prove a singularity theorem with DSNEC as a condition?

Singularity theorem for worldvolume

[Graf, E-AK, Ohanyan, Schinnerl, In preparation]

1. Existence of Cauchy surface Σ
2. Pointwise energy condition $R_{\mu\nu}U^\mu U^\nu \geq -n\kappa$ for all unit timelike U
3. Bound on mean normal curvature $H \leq \beta(\kappa, n, T)$ and $\beta \geq -(n-1)\sqrt{|\kappa|}$
4. Worldvolume bound

$$\int_{\Omega_T} dvol_g R_{\mu\nu}U^\mu U^\nu F(x)^2 \geq -Q_1\|F'\|^2 - Q_0\|F\|^2$$

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- There are generalizations of these theorems with energy conditions averaged over a geodesic segment
- Semiclassical singularity theorems were recently derived with conditions obeyed by some quantum fields for null and timelike geodesics
- For the timelike case we have a singularity theorem with a worldvolume inequality
- Can we have a singularity theorem with DSNEC as the energy condition?