Non-singular spacetimes with the NUT parameter

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05.07.2022

based on:

JL, MO, 2020 Class. Quantum Grav. 37 205007

JL, MO, 2020 Phys. Rev. D 102, 124055

JL, MO, 2021 Phys. Rev. D 104, 024022

Type D spacetimes

In Einstein-Maxwell theory the general Petrov type D Λ-vacuum (possibly with EM-field) spacetime is the Plebański and Demiański spacetime (1976)

$$g = \frac{1}{(1-pr)^2}\left(-\frac{\mathcal{Q}(d\tau - p^2 d\sigma)^2}{r^2+p^2} + \frac{\mathcal{P}(d\tau + r^2 d\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{\mathcal{P}}dp^2 + \frac{r^2+p^2}{\mathcal{Q}}dr^2\right)$$

- $\mathcal{Q}(r)$, $\mathcal{P}(p)$ 4 degree polynomials
- 7 real parameters $\simeq$: cosmological constant, electric & magnetic charges, mass, rotation, acceleration and the NUT parameter
- Limits to Schwarzschild, Kerr, C-metric etc.
- ∂_τ & ∂_σ Killing vector fields

Taub-NUT & Singularities

The Taub-N(ewman)-U(nti)-T(amburino) metric tensor (1963):
l - NUT parameter.

$$ds^2 = -f(r)(dt^\circ + 2l\cos\theta\, d\phi)^2 + \frac{dr^2}{f(r)} + (l^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2} \quad , r_\pm = m \pm \sqrt{m^2 + l^2}$$

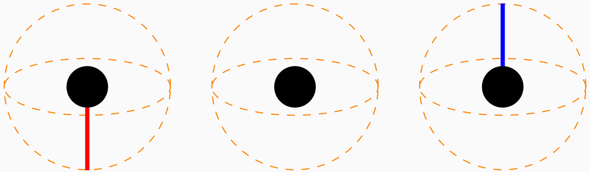
The poles $\theta = 0$ and $\theta = \pi$ are not regular.
CTC around the axes.
But $r = 0$ is! $\implies$ Taub-NUT might by a smooth ”regularizer” of Schwarzschild (?)

Taub-NUT & Misner's interpretation

Solution: Misner glueing (1963) $t^\circ = t + 2l\phi, \quad t^\circ = t' - 2l\phi$

$$ds^2 = -f(r)(dt + 2l(\cos\theta - 1)d\phi)^2 + \frac{dr^2}{f(r)} + (l^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$ds'^2 = -f(r')(dt' + 2l(\cos\theta' + 1)d\phi')^2 + \frac{dr'^2}{f(r')} + (l^2 + r'^2)(d\theta'^2 + \sin^2\theta'\, d\phi'^2)$$



Price: $t = t' - 4l\phi \implies (t, r, \theta, \phi) \in [0, 8\pi l) \times \mathbb{R} \times S^2$

Out goal: extend this construction to spacetimes with Kerr rotation parameter and others

Taub-NUT & interpretations

Cosmological - Taub (1951)

$$g_{t^\circ t^\circ} = -f(r) = -\frac{r^2-2mr-l^2}{r^2+l^2}, \quad g_{rr} = f(r)^{-1}, \quad r_\pm = m \pm \sqrt{m^2+l^2}$$

NUT_-	H_1	TAUB	H_2	NUT_+
∂_t timelike		∂_t spacelike		∂_t timelike

$\Longrightarrow$ finite time, homogeneous cosmological model of Bianchi type IX, without initial or final singularities

Bonnor (1969)

Massless source of angular momentum along the singular axis

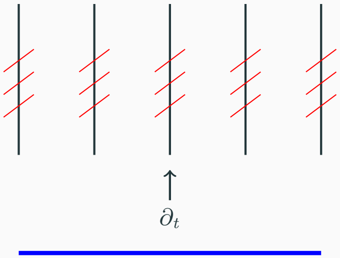
Taub-NUT as a U(1)-principal bundle in Misner's interpretation

Singular at $\theta = \pi$:

$$g = -f(r)(dt + 2l(\cos\theta - 1)\,d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Singular at $\theta = 0$:

$$g = -f(r)(dt' + 2l(\cos\theta + 1)\,d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$



U(1)-principal bundle: $\mathbb{R} \times S^3 \to \mathbb{R} \times S^2$ (extended Hopf fibration)

Kerr-NUT-(anti-) de Sitter

$$
\begin{aligned}
g &= -\frac{\mathcal{Q}}{\Sigma}(dt - A\,d\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(a\,dt - \rho\,d\phi)^2, \\
\Sigma &= r^2 + (l + a\cos\theta)^2, \\
A &= a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta, \\
\rho &= r^2 + (l+a)^2 = \Sigma + aA, \\
\mathcal{Q} &= (a^2 - l^2) - 2mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big), \\
P &= 1 + \frac{4}{3}\Lambda a l\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.
\end{aligned}
$$

a - Kerr rotation parameter.

- $\Sigma = 0$: ring-like singularity, but only for $l^2 < a^2$!
- $A\,d\phi$ is singular at $\theta = \pi$
- $\mathcal{Q}(r_0) = 0$: Killing horizons (up to 4)
- Killing vector fields: ∂_t, ∂_ϕ

General bundle structure

U(1)-principal bundle:

$$\mathbb{R} \times S^3 \xrightarrow{\Pi} \mathbb{R} \times S^2$$

- ξ - generator of $U(1)$-action on $\mathbb{R} \times S^3$
- ω - connection 1-form on $\mathbb{R} \times S^3$
- $f : \mathbb{R} \times S^2 \to \mathbb{R}$
- q - 3D metric tensor on the space of the orbits

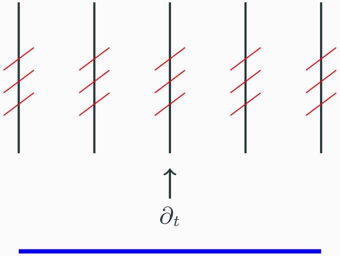
$$g = -(\Pi^* f)\omega \otimes \omega + \Pi^* q, \quad \mathcal{L}_\xi g = 0, \quad \xi^\mu \xi_\mu = -(\Pi^* f)$$

ξ determines all other components!

Kerr-NUT-(anti-) de Sitter as a U(1)-principle bundle

Singular at $\theta = \pi$:

$$g = -\frac{Q}{\Sigma}(dt - A\,d\phi)^2 + \frac{\Sigma}{Q}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta\rho^2 d\phi^2 + \text{smooth}$$



New problem: base space may have conical singularity.
Removable iff $P(0) = P(\pi)$. i.e. $al\Lambda = 0$.

Removing singularities - one step at a time ...

Misner gluing along suitable Killing vector field.

$$\xi = \partial_t + b\, \partial_{\tilde{\phi}}, \qquad b = \text{const.}$$

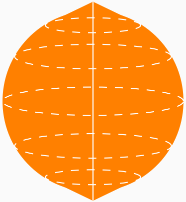
$$\xi(\tau) = 1, \quad \xi(x^i) = 0 \quad , (x^\mu) = (\tau, x^i) = (t,\ r,\ \theta,\ \hat{\phi} := -bt + \tilde{\phi})$$

$$\xi = \partial_\tau$$

$(\tau, r, \theta, \hat{\phi}) \in I_\tau \times \mathbb{R} \times [0, \pi) \times I_{\hat{\phi}}$

... removing the conical singularity ...

Consier q pullbacked to $r =$const.



$$\lim_{\to \text{poles}} \frac{\text{circumference}}{\text{radius}} = 2\pi$$

Continuity condition

$$P(0) = \frac{P(\pi)}{|1 - 4lb|}, \quad P = 1 + \frac{4}{3}\Lambda al \cos\theta + \frac{\Lambda}{3} a^2 \cos^2\theta.$$

The condition is r independent! $\varphi = P(0)\,\hat{\phi}$

Continuity condition

$$P(0) = \frac{P(\pi)}{|1-4lb|}, \quad P = 1 + \frac{4}{3}\Lambda al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.$$

- $l = 0$ - all Killing vectors are fine.
- $l \neq 0$, $a\Lambda = 0 \implies b = 0$ $\xi = \partial_t$.
- $l \neq 0$ - two "principal" Killing vector fields. (but they are equivalent!)

$$b_+ = \frac{2a\Lambda}{3 + a^2\Lambda + 4al\Lambda}, \quad b_- = \frac{3 + a^2\Lambda}{2l(3 + a^2\Lambda + 4al\Lambda)}$$
$$\xi_+ = \partial_t + b_+\partial_{\tilde{\phi}}, \qquad \xi_- = \partial_t + b_-\partial_{\tilde{\phi}}$$

$$\omega = d\tau - \frac{(1-Ab)\Sigma dr + \left(A\mathcal{Q}(1-Ab) - P\sin^2\theta\rho(a-b\rho)\right)\, d\varphi/P(0)}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2}.$$

where $Ad\phi = (a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta)d\phi$ is well defined at $\theta = 0$, fails at $\theta = \pi$.

$$\omega_\varphi(r, \theta = 0) = 0, \quad \omega_\varphi(r, \theta = \pi) = -\frac{4l}{1-4lf}\frac{1}{P(0)} = \text{sgn}(1-4lb)\frac{-4l}{P(\pi)}.$$

$\omega_\phi(r, \theta = \pi)$ is r-independent!

$$\tau = \tau' + \frac{4l}{P(\pi)}\varphi'$$

We have continuity. What about smoothness? Check by inspection.

Cosmological interpretation

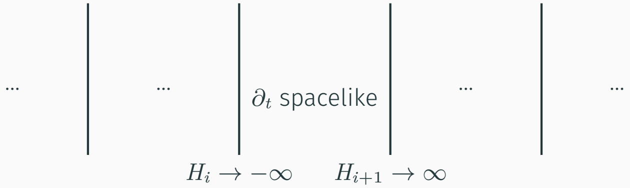
Can we get

$$ds^2 = -\frac{\Sigma}{-\mathcal{Q}}dr^2 + \frac{-\mathcal{Q}}{\Sigma}(dt - Ad\phi)^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adt - \rho d\phi)^2$$

such that $\mathcal{Q} < 0$ and $l^2 > a^2$ (no curvature singularity)?

Yes: for $\Lambda > 0$, sufficiently small m and some inequalities on $a,\ l,\ \Lambda$.

Inhomogeneous (homogeneous for $a = 0$), non-extendable cosmological models without CTC.



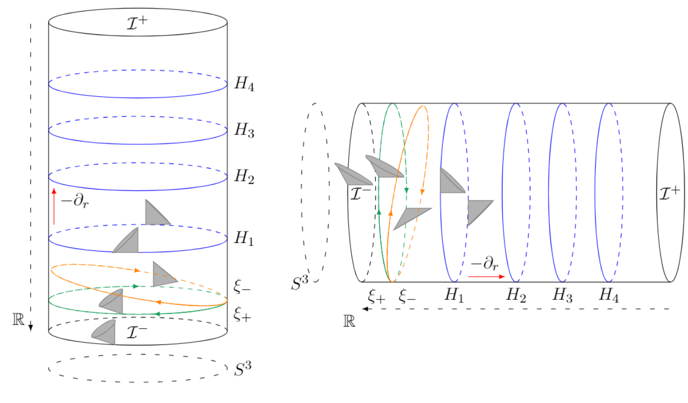
This solution is globally hyperbolic.

(r is globally defined smooth time function $\Rightarrow$ stable causality.

Stable causality and $\overline{J^+(p) \cap J^-(q)}$ is compact $\Rightarrow$ globally hyperbolic)

Geometry of the scri:

$$\frac{\Lambda}{3}\left((1-bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} + P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2$$



Accelerated KN(a)dS

Generalised black hole Plebanski-Demianski solution (Podolsky & Griffiths 2006)
Physical parameters $(a, m, l, \Lambda, \alpha)$. α - acceleration. ω residual gauge.

$$ds^2 = \frac{1}{F^2}\left\{ -\frac{\mathcal{Q}}{\Sigma}(dt - A\,d\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(a\,dt - \rho\,d\phi)^2 \right\}$$

$$F = 1 - \frac{\alpha}{\omega}(l + a\cos\theta)r, \quad P = 1 - a_3\cos\theta - a_4\cos^2\theta,$$

$$\mathcal{Q} = \omega^2 k - 2mr + \epsilon r^2 - 2\frac{\alpha n}{\omega}r^3 - (\alpha k + \frac{1}{3}\Lambda)r^4,$$

$$a_3 = 2\frac{\alpha a m}{\omega} - \alpha^2 a l k - \frac{4}{3}\Lambda a l, \quad a_4 = -\alpha^2 a^2 k - \frac{1}{3}\Lambda a^2,$$

$$\epsilon = \frac{\omega^2 k}{(a^2 - l^2)} + 4\frac{\alpha l m}{\omega} - (a^2 + 3l^2)(\alpha^2 k + \Lambda/3),$$

$$n = \frac{\omega^2 k l}{(a^2 - l^2)} - \frac{\alpha m(a^2 - l^2)}{\omega} + (a^2 - l^2)(\alpha^2 k + \Lambda/3),$$

$$k = \frac{1 + 2\alpha l m/\omega - l^2\Lambda}{3\alpha^2 l^2 + \omega^2/(a^2 - l^2)}.$$

Accelerated and charged KN(a)dS

- $\alpha = 0,\ e \neq 0 \neq g$: no different to uncharged case
- $l = 0$: continuity condition for reduces to $m\alpha = 0$
- $\alpha \neq 0 \neq l$: many results still hold (bundle structure, gluing, equivalence of principal vector fields …). Accelerated spacetimes can be without conical singularity.

Summary

Main results:

- Construction of a globally defined non-singular extension of (accelerated) Kerr-NUT-adS spacetimes
- Extension of Taub cosmological interpretation - inhomogeneus model without initial and final singularities
- Construction of non-singular Killing horizon of Hopf bundle topology (see Thursday A2 17:00-17:30, Killing-Hopf horizons of the Petrov type D, Jerzy Lewandowski)

Thank you for your attention!